**Def** The sectional curvature of \( M \) at \( p \) is \( K(p) : T_pM \times T_pM \to \mathbb{R} \)

\[
K(X,Y) = \frac{\text{Rm}(X,Y,Y,X)}{\|X\|^2\|Y\|^2 - \langle X,Y \rangle^2}
\]

\( K(X,Y) \) only depends on the plane spanned by \( X,Y \).

How to interpret it?

**Prop** Consider the 2-manifold \( S_{xy} = \{ \exp(tX + sY) : |t| < 3, |s| < 3 \} \subset M \).

\( K(X,Y) \) is the scalar curvature of \( S_{xy} \) at \( p \).

**Pf** A radial geodesic \( \gamma \) through \( p \) in \( M \) also lies in \( S_{xy} \). Use \( \sim \) for quantities on \( S_{xy} \).

\[
0 = \nabla_y \dot{\gamma} = \tilde{\nabla}_y \dot{\gamma} + II(\dot{\gamma}, \dot{\gamma})
\]

and the two terms on \( \sim \) separately vanish.

Thus \( II = 0 \). Then \( R = \tilde{R} \), so \( K(X,Y) = \frac{\tilde{R}(X,Y,Y,X)}{\|X\|^2\|Y\|^2 - \langle X,Y \rangle^2} \) as desired.

**Prop** \( K \) determines \( \tilde{R} \).

**Pf** Suppose \( T(X,Y,Y,X) = 0 \) and \( T \) has the symmetries of \( \text{Rm} \).

Then show \( T = 0 \) [Exercise].

**Def** \( M \) has constant curvature \( C \) if \( \forall p \in M, \forall XY \in T_pM \), \( K(X,Y) = C \).

**Prop** \( \mathbb{S}^n \) has const. curv. \( C = 0 \).

\( \mathbb{S}^n_+ \) has const. curv. \( C = 1/R^2 \).

\( \mathbb{H}^n \) has const. curv. \( C = -1/R^2 \).

**Pf** For \( \mathbb{S}^n_+ \) use \( K = \frac{1}{R} / R \)

For \( \mathbb{H}^n \) make a direct computation at a single point [Exercise], then use isometries.