Sobolev spaces

M Riemannian
E orthogonal v.h. over M, with orthogonal connect

Def For \( s \in \mathcal{E}_c(E) \), \( \| s \|_{W^k} = \left( \int_M \| s \|^2 + \| \nabla s \|^2 + \cdots + \| \nabla^k s \|^2 \right)^{1/2} \)

Def \( W^k(E) = \) completion of \( \mathcal{E}_c(E) \) in the norm \( \| \_ \|_{W^k} \). (Banach space)

Bounded embeddings \( \mathcal{E}_c(E) \supset W_1(E) \supset W_2(E) \supset \cdots \supset C_c^\infty(E) \)

\( \subset \) \( L^2(E) \)

Rk \( M \) compact \( \Rightarrow 1) \) \( W^k(E) \) does not depend on the metric in \( M \) or \( E \), nor on \( \nabla \).

(changing these choices gives equivalent norms)

2) choosing a partition of \( 1 \), \( 1 = \sum_i \rho_i \), \( s \mapsto \| s \|_{W^k} \) is equiv to \( s \mapsto \sum_i \| \rho_i s \|_{W^k} \).

Rk Say \( M = S^1 \) of length \( 2\pi \) and \( E \) trivial line bundle; then \( f = \sum_p e^{ipx} \)

has \( \| f \|_{W^k}^2 = 2\pi \sum_p \left( 1 + p^2 + \cdots + p^{2k} \right) |f_p|^2 \)

so \( f \in W^k(E) \iff \sum_p 2^{2k} |f_p|^2 \) converges.

Using Fourier analysis, can define more generally \( W^k(E) \) for any \( s \in \mathbb{R} \) (and any \( M, E \)).

Prop \( k \)-th order differential operator \( D : \mathcal{E}(E) \to \mathcal{E}(F) \)
extends to a bounded \( D : W^k(E) \to W^{k-k}(F) \), \( \forall k \geq k \).

Pf Exercise.

Lemma (Rellich) \( W^1(E) \hookrightarrow W^0(E) \) is compact.

Pf Want to show image of bounded sets is precompact, i.e. any bounded sequence in \( W^1(E) \)
has a subsequence which converges in \( W^0(E) \).
Using partitions of unity and trivializ. reduce to the same question for compactly supported functions on $\mathbb{R}^n$. This is the same question for compactly supported $f$'s on $T^n = (S^1)^n$.

$$f(x) = \sum_p \hat{f}_p e^{ipx}, \quad p = (p_1, \ldots, p_n) \in \mathbb{Z}^n$$

$$\|f\|_{\mathcal{W}_1}^2 = \sum_p (1 + \|p\|^2) |\hat{f}_p|^2$$

Fix $\varepsilon > 0$.

Let $\mathcal{Z}_N = \{f \in \mathcal{W}_1(T^n); \hat{f}_p = 0 \text{ for } \|p\| \leq N\}$ "high-frequency part"

$$f \in \mathcal{Z}_N \text{ and } \|f\|_{\mathcal{W}_1} < 1 \Rightarrow \|f\|_{\mathcal{W}_0} < \frac{1}{1 + N^2}. \text{ Fix } N \text{ s.t. } \frac{1}{1 + N^2} < \frac{\varepsilon}{\sqrt{2}}.$$  

Meanwhile $\mathcal{Z}_N^\perp = \{f \in \mathcal{W}_1(T^n); \hat{f}_p = 0 \text{ for } \|p\| > N\}$ is finite-dim., so

$$\{f: f \in \mathcal{Z}_N^\perp \text{ and } \|f\|_{\mathcal{W}_1} < 1\} \text{ can be covered by finitely many } \|\cdot\|_{\mathcal{W}_0}-\text{balls of radius } < \frac{\varepsilon}{\sqrt{2}}.$$  

Thus $\{\|f\|_{\mathcal{W}_1} < 1\} \text{ can be covered by finitely many } \|\cdot\|_{\mathcal{W}_0}-\text{balls of radius } < \varepsilon.$

So it's totally bounded subset of a complete metric space $\Rightarrow$ precompact (Heine-Borel)

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**Lemma (Sobolev)**: If $k - \frac{n}{2} > k$, then $W_k(E) \hookrightarrow C^k(E)$.

**Pf**: For $k = 0$:

Use partitions of unity to reduce to $f$'s supported inside unit ball $B \subset \mathbb{R}^n$.

Assume $f$ smooth, then $f(0) = C \int_0^1 r^{n-1} \frac{\partial f}{\partial r} \left[ f \text{ by parts in a fixed radial dir}\right]$  

i.e. $f(0) = C \int_B \frac{\partial f}{\partial r} r^{n-1} \text{ vol} \left[ \text{average over all radial dir}\right]$
\[ |f(x)| \leq C \sqrt{\int_B \left| \frac{\partial f}{\partial r} \right|^2 \text{vol} \sqrt{\int_B r^{2(l-w)} \text{vol}}} \quad \text{[Cauchy–Schwarz]} \]
\[ \leq C' \|f\|_{W_{l}} \quad \text{[}\lambda \geq \frac{\lambda}{2}\text{]} \]

Therefore \( \|f\|_{L^\infty} \leq C' \|f\|_{W_{l}} \)

and \( C^0(E) \) is the completion of \( E(E) \) in the \( L^\infty \) norm (uniform limit of continuous functions is continuous)

Similar for \( k > 0 \), just "differentiate under the \( \int \) sign" in the above.

We also need:

**Def** \( s \in E(E) \) if \( \|s\|_{W_{l}} = \min \{ C : \langle s, s' \rangle_{L^2} \leq C \|s\|_{W_{l}} \|s'\|_{W_{l}} \forall s' \in W_{l}(E) \} \)

**Def** \( W_{-l}(E) \) = completion of \( E(E) \) wrt \( \|\cdot\|_{W_{l}} \)

**Rk** elements of \( W_{-l}(E) \) may be distributions which are not functions.

**Eg** if \( M=S^1 \), \( E \) thr., \( \delta(x) \in W_{-l}(E) \) [since we showed already that \( \|f\|_{L^\infty} \leq C \|f\|_{W_{l}} \)]

**Prop** \( W_{-l}(E) \cong W_{l}(E)^* \) [space of bounded linear operators on \( W_{l}(E) \)]

**Pf** Cauchy sequence \( \{s_n\} \in E(E), s_n \rightarrow s \) in \( W_{-l}(E), s' \in W_{l}(E) \):
- then \( \langle s_n, s' \rangle_{L^2} \) is also Cauchy sequence, call its limit \( \langle s, s' \rangle \).
- This gives \( W_{-l}(E) \rightarrow W_{l}(E)^* \). To see it's \( \cong \), need to know that this pairing is nondegenerate: just use the fact that it's nondegenerate on \( E(E) \) and \( E(E) \) is dense in both \( W_{l}(E) \) and \( W_{-l}(E) \).

**Prop** \( \varphi_{l}(E) \hookrightarrow \varphi_{-l}(E) \) is bounded.

**Pf** \( \langle s, s' \rangle_{L^2} \leq \|s\|_L \|s'\|_L \leq \|s\|_{W_{l}} \|s'\|_{W_{-l}} \Rightarrow \|s\|_{L^2} \geq \|s\|_{W_{l}} \).
Prop

Any 2nd-order diff. op extends to a bounded map \( W_0(E) \to W_1(E) \).

Pf

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