Recall that if \( E \) is any bundle over \( M \), and we fix a local basis \( \{ e_i \} \), a connection \( \nabla \) can be represented by 1-forms \( A^j_i \) characterized by \( \nabla e_i = A^j_i e_j \).

Under a change of basis \( e_i = C^j_i e_j \), we have \( A^j_i' = (C^j_i)' A^j_i C_i^k + (C^i_j)' dC_i^k \).

Less index version: \( A \in \mathcal{E}(T^* \text{End}(\mathbb{R}^n)) \) with \( A' = C^{-1} dC + C^{-1} dC \).

Letting \( G = \text{GL}(n, \mathbb{R}), \mathfrak{g} = \text{End}(\mathbb{R}^n) \), so this is also \( A \in \mathcal{E}(T^* M \otimes \mathfrak{g}) \).

Now we want to give a global interpretation of this formula. Let \( P \) be the bundle of frames in \( E \). Over basis \( \{ e_i \} \) then \( \to \) a section \( s \) of \( P \). \( s: M \to P \).

Want to find a single 1-form \( \omega \in \mathcal{E}(T^* P \otimes \mathfrak{g}) \) such that \( A = s^* \omega \).

Transformation law above then relates \( s^* \omega \) and \( (s^\tau)^* \omega = s^* (\hat{C}^* \omega) \) for \( \tau: M \to G \) inducing \( \hat{C}: P \to P \).

It says, \( \hat{C}^* \omega = C^{-1} \omega C + \pi^* [C^{-1} dC] \)

Thus:

**Def/Prop** \( P \) principal \( \text{GL}(n, \mathbb{R}) \)-bundle:

A connection in \( P \) is a 1-form \( \omega \in \mathcal{E}(T^* P \otimes \mathfrak{g}) \) such that

1) \( \forall C: M \to G \), \( \hat{C}^* \omega = C^{-1} \omega C + \pi^* [C^{-1} dC] \)

or equivalently,

2a) \( \omega(\pi^* Y) = \text{Ad}(C^i_j) \omega(Y) \quad \forall g \in G, \ Y \in TP \)

2b) \( \omega(\sigma(X)) = X \quad \forall X \in \mathfrak{g} \).

**Pf**

1) \( \Rightarrow \) 2a): take \( C \) constant.

1) \( \Rightarrow \) 2b): \( \forall X, Y \in TP \) with \( \pi^* (\pi^* Y) = m, \pi^* Y \neq 0 \). \( \forall X \in \mathfrak{g} \).

Consider \( C: M \to G \) with \( C(m) = I \), \( dC(\pi^* Y) = X \).

Then, \( \hat{C} Y Y = [\sigma(X)](p) \) (see behavior of \( \hat{C} Y \))

But \( (\hat{C}^* \omega)(Y) - \omega(Y) = (\pi^* dC)(Y) \) by 1),

ie \( \omega(\hat{C} Y - Y) = X \).
2) $\Rightarrow$ 1):

\[
\begin{align*}
\text{Def: } P \text{ principal } G\text{-bundle:} \\
\text{A connection on } P \text{ is a 1-form } \omega \in \mathcal{E}(T^* P \otimes \mathfrak{g}) \text{ such that} \\
\omega(g_*^* Y) &= \text{Ad}(g^{-1}) \omega(Y) \quad \forall g \in G, \ Y \in TP, \\
\omega(\sigma(X)) &= X \quad \forall X \in \mathfrak{g}.
\end{align*}
\]

\[
\begin{align*}
\text{Def: } T^\text{horz} P &= \ker \pi^* \theta. \text{ Given a conn. in } P, \text{ define } T^\text{horz} P &= \ker \omega. \\
\text{Prop:} & \\
1) & T^\text{vert} P = T^\text{horz} P \oplus T^\text{horz 2} P. \\
2) & g_*^* (T^\text{horz 2} P) = T^\text{horz 2} P. \\
3) & \text{Any distribution } T^\text{horz 2} P\text{ obeying } 1,2) \text{ determines a connection in } P.
\end{align*}
\]

\[
\begin{align*}
\text{Rk: } \text{Given a connection in } P \text{ we also obtain horiz. distib. on all } Y^*_p.
\end{align*}
\]

\[
\begin{align*}
\text{Def: } O(n) = \{ M \in G(n): M^T M = 1 \}, \quad SO(n) = \{ M \in O(n): \det M = 1 \}.
\end{align*}
\]

\[
\begin{align*}
\text{Ex: } G = SO(2): \text{ } P \text{ is a circle bundle over } M. \Rightarrow \text{wrt a local trivialization, } P \text{ have coords } (x_i^*, \Theta). \\
\text{Then identifying } G\text{ with } T^*_P, \omega \text{ is just a 1-form}, \\
\omega = \omega_o + d\Theta \quad \text{where } \omega_o(\Theta_{\Theta}) = 0.
\end{align*}
\]
If $E$ is an orthonormal bundle, we can also consider the bundle of orthonormal frames in $E$, $P_x = \{\text{orthonormal bases for } E_x\}$. $P_x$ is a principal $O(n)$-bundle.

Given a homomorphism $\rho: G \rightarrow H$, $G$ acts on $H$ (by $g \cdot h = \rho(g)h$).

Then if $P$ principal $G$-bundle, $H$ acts on $P \times_G H$ by $[(p, h)] \mapsto [(p, \rho(h))]

This makes $P \times_G H$ into a principal $H$-bundle.

If $Q$ is an $H$-bundle realized as $P \times_G H$ for some $G \subset H$, say $Q$ "reduces to $G$".

Say $Q$ is bundle of frames in some $E$. Fix a metric on $E$. Then let $P$ be bundle of orh. frames in $E$. $P \times_{O(n)} G(\mathbb{R}) \sim Q$ [Exercise].

Thus $Q$ reduces to $O(n)$.

But in general, given $H \subset G$, there are topological obstructions to a $G$-bundle reducing to $H$. (e.g. consider $H = \{1\}$)

---

**Curvature**

Given principal $G$-bundle $P \rightarrow M$ and conn. $\omega$ on $P$,

define its curvature $\Omega = d\omega + \omega \wedge \omega \in \mathfrak{g}(\Lambda^2 T^*P \otimes \mathfrak{g})$

i.e. $\Omega(Y, Z) = d\omega(Y, Z) + [\omega(Y), \omega(Z)] \quad Y, Z \in TP$

**Prop** $\Omega$ obeys

1. $\Omega(g_*Y, g_*Z) = \text{Ad}(g^{-1})\Omega(Y, Z) \quad \forall g \in G, Y, Z \in TP$

2. $\Omega(Y, Z) = 0 \quad \forall Y \in (TP)_{vol}, Z \in TP$

**Pf**

1) from the analogous property for $\omega$.

2) $\Omega(Y, Z) = Y\omega(Z) - Z\omega(Y) - \omega([Y, Z]) + [\omega(Y), \omega(Z)]$

If $Y, Z$ both vertical then extend them to $Y = \sigma(W), Z = \sigma(X)$ for $W, X \in \mathfrak{g}$.
Then $\Omega(\gamma, Z) = \sigma(\omega) \cdot X - \sigma(X) \cdot \omega - \omega([\sigma(\omega), \sigma(X)]) + [\omega, X]$

\[
\begin{align*}
&= -[\omega, X] + [\omega, X] = 0
&\text{using } [\sigma(\omega), \sigma(X)] = \sigma([\omega, X])
&\text{(This is essentially Maurer-Cartan equation)}
\end{align*}
\]

If $Y$ vertical and $Z$ horizontal, then extend $Y = \sigma(\omega)$ and $Z$ as section of ker $\omega$:

then $\Omega(Y, Z) = \sigma(\omega) \cdot w(Z) - Z \cdot \omega - \omega([\sigma(\omega), Z]) + [\omega, \omega(Z)]$

\[
\begin{align*}
&= -[\omega, \omega(Z)] = 0
&\text{using } [\sigma(\omega), \omega(Z)] = \omega([\omega, Z])
&\text{(Exercise)}
\end{align*}
\]

Cor: If $s, s'$ are two sections of $P$ with $s' = sg$ for $g : M \to G$,

then $s'^*\Omega = Ad(g^{-1}) s^*\Omega$

Pf: Just as we did for $\omega$, using the preceding prop.

**Rk:** $\mathfrak{g}_P = \mathfrak{p} \times \mathfrak{g}/(\mathfrak{p}, X) \sim (P, Ad(g^{-1}) X)$

$\Omega$, induces a section $F \in \mathfrak{e}(\Lambda^2 T^* \otimes \mathfrak{g}_P)$, by $F(X, Y) = [(p, \Omega(\hat{X}, \hat{Y}))]$

for any $\hat{X}, \hat{Y} \in T_p P$, with $\pi_* \hat{X} = X, \pi_* \hat{Y} = Y$

Fix any rep $\rho : G \to \text{Aut}(V)$.

Prop

1) $\omega$ induces a connection $\nabla$ in $V_P$, by $\nabla[(s, v)] = [(s, dv + (s^*v))]$

2) Its curvature $F_\nabla \in \mathfrak{e}(\Lambda^2 T^* \otimes \text{End} V_P)$ is $F_\nabla(X, Y)[(s, v)] = [(s, \Omega(\hat{X}, \hat{Y}) v)]$

for any $\hat{X}, \hat{Y} \in T_p P$ with $\pi_* \hat{X} = X, \pi_* \hat{Y} = Y$.

Pf: Exercise.

**Rk** Conversely, if $\rho$ is faithful, a conn. in $V_P$ induces one in $P$. 