First, fill in a bit more about holonomy in principal bundles

\( M \) smooth sfd, \( P \) principal \( \mathcal{G} \)-bundle over \( M \), \( \omega \) connection in \( P \).

**Def/Prop:** Given a path \( \gamma : [0, 1] \to M \) and \( p \in P_{\gamma(0)} \), the lift \( \tilde{\gamma}_{p} : [0, 1] \to P \) is the unique map with \( \tilde{\gamma}_{p}(0) = p \), \( \tilde{\gamma}_{p}'(t) \in \ker \omega \forall t \).

**Def:** \( \text{P}_{\gamma} : P_{\gamma(0)} \to P_{\gamma(1)} \)

\[ p \mapsto \tilde{\gamma}_{p}(1) \]

**Prop:** \( \text{P}_{\gamma} \circ \gamma \cdot \gamma(p) = \text{P}_{\gamma} \gamma(pg) \).

**Pf:** \( \tilde{\gamma}_{pg}(t) = \tilde{\gamma}_{p}(t) \gamma_{p}(t) \), since it indeed has \( \tilde{\gamma}_{pg}(0) = pg \) and \( \tilde{\gamma}_{pg}'(t) = g \tilde{\gamma}_{p}'(t) \in g \ker \omega = \ker \omega \).

**Def:** \( \text{Aut } P_x = \{ \gamma_x : P_x \to P_x \ s.t. \ \gamma_x(p)g = \gamma_x(pg) \} \).

\( \text{Aut } P = \bigcup_x \text{Aut } P_x \)

**Prop:** \( \text{Aut } P = P \times \mathcal{G} \) where \( \mathcal{G} \) acts on itself by adjoint action, \( \text{Ad}_g(g') = gg'g^{-1} \)

**Pf:** \( \gamma \mapsto \{ (p, g) : \gamma(p) = pg \} \)

well defined b/c \( \gamma(pg) = (pg)(g'\gamma gg') \)

and check it's bijective...

**Rk:** In any local trivial \( P \), \( \text{Aut } P \) just acts as left multiplication by \( \mathcal{G} \).

**Def:** \( L_xM = \{ \gamma : [0, 1] \to M, \ \gamma(0) = x(1) = x \} \)

**Cor:** \( \gamma \in L_xM \Rightarrow \text{P}_{\gamma} \in \text{Aut } P_x \)

**Def:** \( \text{Hol}_x \omega = \{ \text{P}_{\gamma} : \gamma \in L_xM \} \subset \text{Aut } P_x \)

\( \text{Hol}_x^0 \omega = \{ \text{P}_{\gamma} : \gamma \in L_xM, \ \gamma \text{ homotopic to trivial loop} \} \subset \text{Aut } P_x \).
Prop \( M \text{ connected } \Rightarrow \text{Hol}_x \omega = \text{Hol}_{x'} \omega \ \forall x, x' \in M. \)

Pf \( \frac{Q_x}{\gamma} \xrightarrow{\gamma^{-1}} Q_{x'}^{y, \gamma^{-1}} \) \( P y y y^{-1} = \overline{P_y} \circ \overline{P_y} \circ \overline{P y^{-1}} \) so \( \overline{g} \overline{\gamma} = \overline{P_y} \circ \overline{g} \circ \overline{P y^{-1}} \) does the job

\( \text{Hol}_x \omega \ \text{Hol}_{x'} \omega \)

Alternatively: Def \( \text{Hol}_p \omega = \{ g : pg \text{ can be reached from } p \text{ by parallel transport} \} \subset G \)

Prop \( M \text{ connected } \Rightarrow \text{all Hol}_p \omega \text{ and all Hol}_x \omega \text{ are isomorphic.} \)

Pf Exercise.

If \( \text{Hol}_{x} \omega \not\cong \text{Aut} P_\omega \), then \( P \) can be reduced to a smaller structure group:

Thm Let \( H = \text{Hol}_p \omega \). Define \( Q = \{ q \in P : q \text{ can be reached from } p \text{ by parallel transport} \} \)

Then \( Q \) is a principal \( H \)-bundle and \( P = Q \times_H G \).

Pf \( H \) acts on \( Q \) by \( \underline{q} = \overline{P_y} \circ \underline{p} \) \( \Rightarrow \underline{q} h = \overline{P_y} \circ \underline{h} \).\( \overline{q} \).\( \overline{h} \).\( \overline{p} \).

(Well defined since if \( \overline{P_y} \circ \underline{p} = \overline{P_y} \circ \underline{q} \) then \( \overline{P_y} = \overline{P_y} \) so in fact
\( \overline{P_y} \circ \underline{h} = \overline{P_y} \circ \underline{h} \)
and \( Q/H = M \) [Exercise].

The desired map \( Q \times_H G \to P \) is \( (\overline{P_y} \circ \underline{p}, g) \to \overline{P_y} \circ \underline{p} \).

Ex For \( M \) Riemannian and \( P = \text{bundle of frames} \), fix some \( e \in P_x \).

Then \( Q = \text{all frames related to } e \) by parallel transport.

If \( e \) is an orthogonal frame, \( \text{Hol}_e \omega \subset O(n) \).

So \( P \) can be reduced to \( O(n), \) or smaller. [We already knew this, but now see it as a corollary of the holonomy gp.]

Ref We didn't just get an abstract reduction, but something more concrete:

Def A \( G \)-structure on \( M \) is a principal \( G \)-subbundle \( Q \subset P \).

What other reductions can we get in this way?
The possibilities have been classified:

**Def.** $M$ is a **Riemannian symmetric space** if $\forall x \in M, \exists g_x \in Isom(M)$ s.t. $g_x(x) = x$,

$g_{x^*} = -1$ on $T_x M$.

**Prop.** $M$ is Riem. sym $\iff$ $M$ is homogeneous and $\exists x \in M, g_x \in Isom(M)$ s.t. $g_x(x) = x$,

$g_{x^*} = -1$ on $T_x M$.

**Def.** $M$ is locally **Riemannian symmetric space** if $\forall x \in M, \exists$ an nbhd of $x$ which is isometric to an open subset of a Riemann symmetric space.

**Def.** $M$ is reducible if $M \cong (M_1 \times M_2)/\Gamma$ for some $M_1, M_2$ and $\Gamma \subset Isom(M_1 \times M_2)$ finite.

Irreducible otherwise.

**Thm.** [Borel]: If $M$ is not locally symmetric, irreducible and simply connected, holonomy gp is $\cong$ one of the following:

- **Riemannian** $\text{SO}(n)$
- **Kähler** $\text{U}(n) / \text{U}(n/2)$
- **Calabi-Yau** $\text{SU}(n) / \text{SU}(n/2)$
- **Hyperkähler** $\text{Sp}(n/4)$
- **Quatermion-Kähler** $\text{Sp}(n/4) : \text{Sp}(1)$
- **$G_2$** 7-manifolds only!
- **Spin(7)** 8-manifolds only!

Here, $\text{Sp}(k) = \{ g \in GL(k, HH) : \langle x, y \rangle = \langle gx, gy \rangle \}$

with $\langle \cdot, \cdot \rangle$ the standard pairing on $HH^k$

e.g. $\text{Sp}(1) \cong \text{SU}(2)$

- $G_2 = \{ \text{automorphisms of } D \}$ s $\dim_{\mathbb{R}} G_2 = 14$; $G_2$ has 7-dim irrep.