Derivatives

What is the tangent line to the graph \( y = x^2 \) at the point \((1, 1)\)?

The derivative of a function \( f \) at a number \( a \) is

\[
f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
\]
If \( f(x) = 7x^2 - 3x + 1 \) what is \( f'(x) \)?

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{(7(x+h)^2 - 3(x+h) + 1) - (7x^2 - 3x + 1)}{h}
\]

\[
= \lim_{h \to 0} \left( \cdots \right) = \lim_{h \to 0} \frac{14xh - 3h + 7h^2}{h} = \lim_{h \to 0} \frac{14x - 3 + 7h}{1} = 14x - 3
\]

What is the tangent line to the graph of \( y = f(x) \) at \((-1, 11)\)?

Slope is \( f'(-1) = 14(-1) - 3 = -17 \)

Line w/slope -17 thru (-1, 11) is

\[
y - 11 = -17(x + 1)
\]

\[
y = -17x - 6
\]

Where does the graph of \( y = x^2 \) have a horizontal tangent line?

i.e. if \( f(x) = x^2 \), where does \( f'(x) = 0 \) ? (horizontal \( \iff \) slope = 0)

\[
f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \cdots = 2x
\]

So \( f'(x) = 0 \) only at \( x = 0 \).

\[2x = 0\]

If \( f(x) = \frac{1}{x^2} \) what is \( f'(x) \)?

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{(x+h)^2} - \frac{1}{x^2}
\]
\[
\lim_{h \to 0} \frac{x^2 - (x+h)^2}{h} = \lim_{h \to 0} \frac{x^2 - (x+h)^2}{h \cdot (x+h)^2 \cdot x^2} = \lim_{h \to 0} \frac{x^2 - (x^2 + 2xh + h^2)}{h \cdot (x+h)^2 \cdot x^2} = \lim_{h \to 0} \frac{-2xh}{h \cdot (x+h)^2 \cdot x^2} = \lim_{h \to 0} \frac{-2x}{(x+h)^2 \cdot x} = \frac{-2x}{x^4} = \frac{-2}{x^3}
\]

Check: for \(x > 0\) slope of tangent line is negative, so should have \(f'(x) < 0\) and indeed, \(-\frac{2}{x^3} < 0\).

**Ex.** If \(f(x)\) is

\[y = \frac{1}{x^2}\]

Sketch \(f'(x)\),

\[y = f'(x)\]
Ex. If \( f(x) = \sqrt{x} \)

1. Sketch \( f'(x) \)

\[ f(x) = \sqrt{x} \]

\[ f'(x) = \frac{1}{2\sqrt{x}} \]

2. Calculate \( f'(x) \).

\[ f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \]

\[ = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \]

\[ = \lim_{h \to 0} \frac{(x+h) - x}{h \cdot (\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{h}{h \cdot (\sqrt{x+h} + \sqrt{x})} \]

\[ = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \]
We say $f(x)$ is **differentiable** at $x$ if $f'(x)$ exists, i.e. $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ exists, not $\pm \infty$.

**Example** $f(x) = x^2$ is differentiable at all real $\# x$. ($\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = 2x$)

**Example** Where is $f(x) = |x|$ differentiable?

For $x > 0$, $f'(x) = 1$  
$x < 0$, $f'(x) = -1$  
$x = 0$, $f'(x) = \lim_{h \to 0} \frac{|h|}{h}$  
$= \lim_{h \to 0} \frac{|h|}{h} = \lim_{h \to 0} \frac{|h|}{h}$

so $f(x)$ is differentiable at all $x$ except $x = 0$.

(Generally: **sharp point** $\rightarrow$ **not differentiable**)

**Example** $f(x) = \begin{cases} 1 & x \neq 0 \\ 2 & x = 0 \end{cases}$

For $x \neq 0$, $f'(x) = 0$  
For $x = 0$, $f'(x) = \lim_{h \to 0} \frac{f(h)-f(0)}{h}$

for $h$ small: $\frac{f(h)-f(0)}{h} = \frac{1-2}{h} = -\frac{1}{h}$

so $\lim_{h \to 0}$ DNE.
So \( f(x) \) is not differentiable at \( x=0 \).

General rule: if \( f \) is not continuous, then it’s not differentiable.

Ex

Differentiable at all \( x \) except at \( x = -6, -3, 1, 2, 7, 11 \)

Interpretation of \( f'(x) \)

If \( x(t) \) is the position of an object at time \( t \) seconds (in meters), then \( x'(t) \) is the velocity of the object at time \( t \) (in meters/sec).

Ex

An electron in a uniform electric field moves as

\[
x(t) = \frac{1}{2} t^2
\]

What is its velocity at time \( t \)?

\[
v(t) = x'(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h} = \lim_{h \to 0} \frac{\frac{1}{2} (t+h)^2 - \frac{1}{2} t^2}{h}
\]

\[
= \ldots = \frac{t}{2}
\]

Is it speeding up or slowing down? Speeding up.
In general, if \( t = \text{time} \)
then \( f'(t) \) is the rate of change of \( f(t) \).

Ex. if \( V(t) \) is the volume of water in Lake Travis (in gal)
at time \( t \) (in sec)
then \( V''(t) \) is the rate water is entering/leaving the lake (in gal/sec).