Lecture 8

17 Sep 2018

My office hour: M 2-3 ← today
W 3:30-4:30 ← new
in RLM 9.134

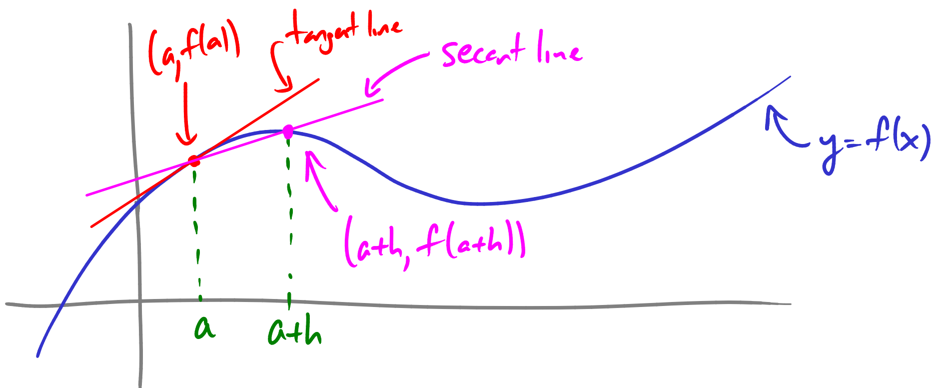
Other resources:
Calc Lab
TA office hrs
Other students!

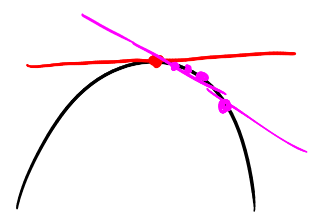
Last time: limits as $x \to \pm\infty$

Ex
$$\lim_{x\to\infty} \frac{3x^4+7x^2+7}{9x^4+8x} = \lim_{x\to\infty} \frac{3x^4}{9x^4} = \lim_{x\to\infty} \frac{3}{9} = \frac{1}{3}$$

$$\lim_{x\to-\infty} \frac{x^2+1}{2x} = \lim_{x\to-\infty} \frac{x^2}{2x} = \lim_{x\to-\infty} \frac{x}{2} = -\infty$$

Derivatives





idea: define the tangent line to $y=f(x)$ at $(a,f(a))$ as $h\to 0$ limit of the secant lines through $(a,f(a))$ and $(a+h, f(a+h))$

slope of secant line: $\frac{\text{rise}}{\text{run}} = \frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$

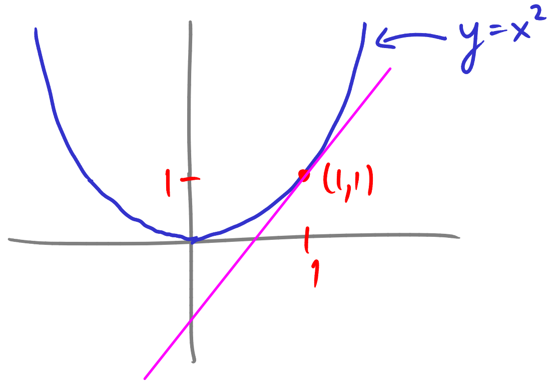
So, slope of tangent line at $(a,f(a))$ is $\lim_{h\to 0}$ (slope of secant line)

$$= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$ (if this limit exists)

We call this slope the derivative:

so, say derivative of f at a point a is
$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$ if it exists.

Q What is the tangent line to the graph $y=x^2$ at $(1,1)$?



$$\text{slope} = f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^2-1^2}{h} \quad \frac{0}{0}$$

$$= \lim_{h\to 0} \frac{1+2h+h^2-1}{h} = \lim_{h\to 0} \frac{2h+h^2}{h} = \lim_{h\to 0} 2+h = 2.$$

So, the tangent line goes through $(1,1)$ and has slope 2: so it's

$$y-1 = 2(x-1) \quad \text{ie} \quad y-1 = 2x-2$$
$$\underline{y = 2x-1}$$

Ex If $f(x) = 7x^2-3x+1$

① what is $f'(x)$?

② what is the tangent line to the graph $y=f(x)$ at $(x,y) = (-1,11)$?

① $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(7(x+h)^2-3(x+h)+1)-(7x^2-3x+1)}{h}$$

$$= \cdots = \lim_{h\to 0} \frac{14xh-3h+7h^2}{h} \quad \frac{0}{0}$$

$$= \lim_{h\to 0} 14x-3+7h$$

$$= \underline{14x-3}$$

"point-slope formula"
$y-y_0 = m(x-x_0)$

② slope is $f'(-1) = 14(-1)-3 = -17$

line with slope -17 thru $(-1,11)$ is $y-11 = (-17)(x-(-1))$

$\vdots$

$$\underline{y = -17x-6}$$

Ex When does the graph of $y = x^2 + 7$ have a horizontal tangent line?

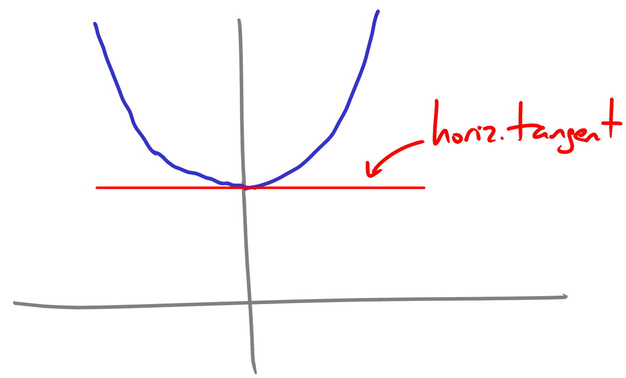
Horizontal tangent $\longleftrightarrow$ slope of tangent line $= 0$

So, need to solve $f'(x) = 0$

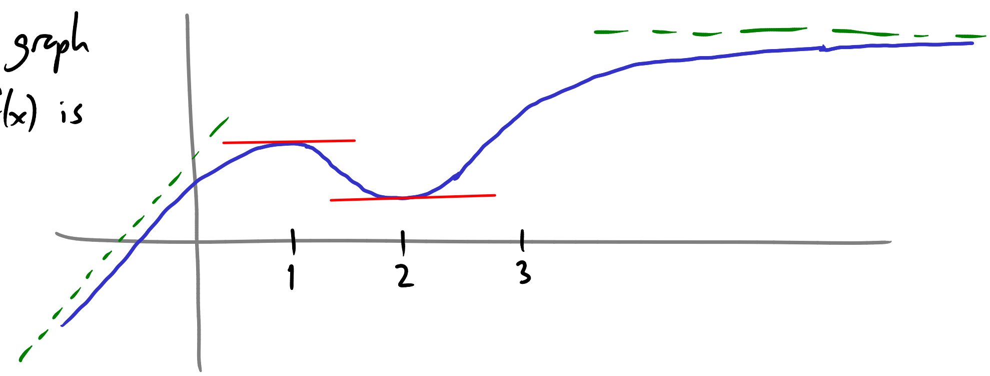
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left((x+h)^2 + 7\right) - (x^2+7)}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} 2x + h = 2x$$

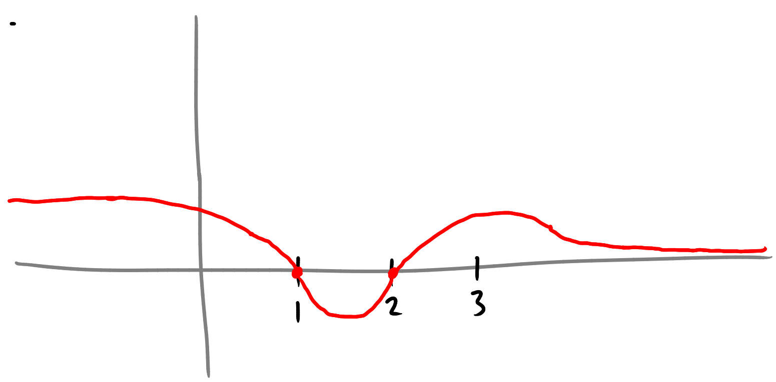
So $f'(x) = 0$ when $2x = 0$ ie $\underline{x = 0}$



Q If the graph of $y = f(x)$ is

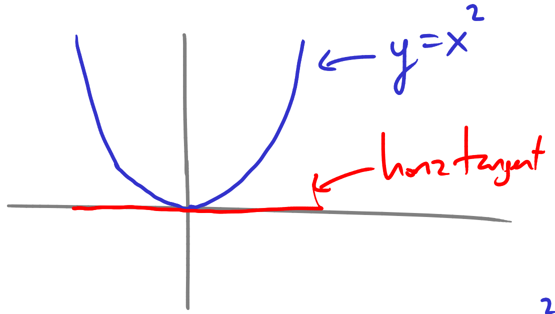


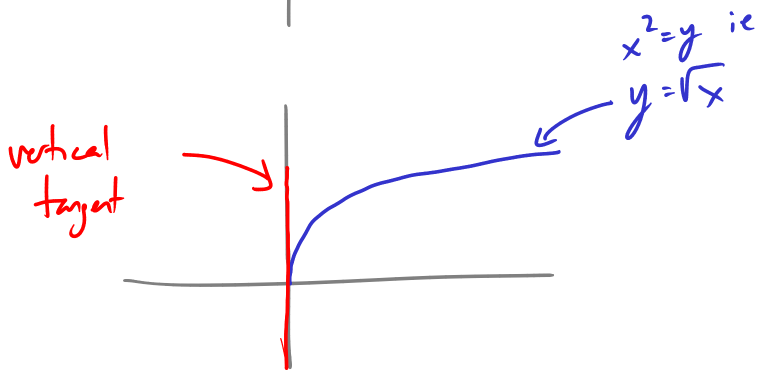
sketch graph of $y = f'(x)$.



Ex If $f(x)=\sqrt{x}$

① sketch $f'(x)$.

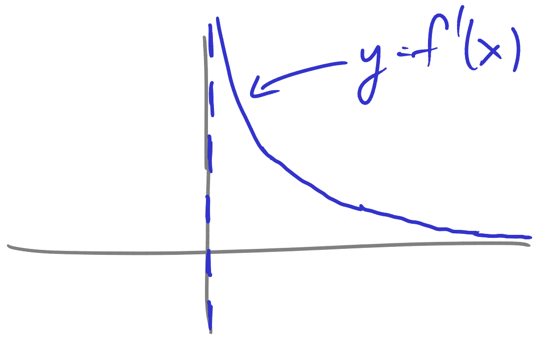




vertical tangent at $x=0$

$\Rightarrow$ slope of tangent line is ∞ at $x=0$

so $f'(x)$ should go to ∞ as $x\to 0$



② calculate $f'(x)$. $f(x)=\sqrt{x}$

$$f'(x)=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$=\lim_{h\to 0}\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$=\lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x}}$$