Exam result: ~ 87%
Office hr today (T) 4-5:30
Next HW due Fri night (Sat morning) 3am

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Last time: implicit differentiation
derivatives of logarithmic and inverse trig functions

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Exponential growth and exponential decay

Suppose we have a function \( P(t) \) such that \( \frac{dP}{dt} \) is proportional to \( P \) itself.

i.e.
\[
\frac{dP}{dt} = kP
\]
for some constant \( k \).

Then what is \( P(t) \)?

One possibility:
\[
P(t) = e^{kt}
\]
\[
\frac{dP}{dt} = ke^{kt} = kP
\]
\[
P(t) = 0
\]
\[
\frac{dP}{dt} = 0 = k \cdot 0 = kP
\]

\[
P(t) = \left( \frac{at}{ln a} \right)
\]
\[
\frac{d}{dt} P(t) = a t \cdot \frac{ln a}{ln a} = a t = (ln a) \cdot \frac{a t}{ln a}
\]
\[
= (ln a) \cdot P
\]

Most general possibility:
\[
P(t) = Ce^{kt}
\]

\[
\text{[then } \frac{dP}{dt} = C \cdot ke^{kt} = k \cdot Ce^{kt} = kP(t) \text{]}
\]

So:
\[
\frac{dP}{dt} = kP(t) \implies P(t) = Ce^{kt} \text{ for some } C.
\]
Examples:
1. Population growth under constant conditions \( (k > 0) \)
2. Radioactive decay \( (k < 0) \)
3. Compound interest

Ex
A population of bacteria grows from 1 g at 2 pm to 15 g at 5 pm.
What will be the mass of bacteria at 10 pm?

\[ P(t) = Ce^{kt} \]

\[ P(0) = 1 \Rightarrow Ce^{k0} = 1 \]
\[ C = 1 \]

\[ P(3) = 15 \Rightarrow Ce^{k3} = 15 \]
\[ e^{k3} = 15 \]
\[ 3k = \ln 15 \]
\[ k = \frac{\ln 15}{3} \]

We want \( P(8) \).

\[ P(8) = Ce^{k8} = 1 \cdot e^{\left(\frac{\ln 15}{3}\right)8} = e^{\frac{8}{3} \ln (15)} \]

\[ \approx 1368 \text{ g bacteria} \]

Ex
The half-life of radium-226 is 1590 yrs.
Suppose we have 100 mg of radium-226.
When will it be reduced to 30 mg?

\[ P(t) = Ce^{kt} \]
\[ P(0) = 100 \Rightarrow Ce^{k0} = 100 \]
\[ C = 100 \]
\[ \frac{1}{2} \text{-life means the amount of time it takes } P(t) \text{ to be reduced by } \frac{1}{2}. \]

i.e. \[ \frac{1}{2} \text{-life is } 1590 \text{ yrs} \Rightarrow e^{k \cdot (1590)} = \frac{1}{2} \quad t \text{ in yrs} \]

\[ k \cdot 1590 = \ln \left( \frac{1}{2} \right) \]

\[ k = \frac{\ln \left( \frac{1}{2} \right)}{1590} = -\frac{\ln 2}{1590} \]

Now need to find \( t \) such that

\[ P(t) = 30 \]

\[ Ce^{k \cdot t} = 30 \]

\[ 100 e^{-\frac{\ln 2}{1590} \cdot t} = 30 \]

\[ e^{-\frac{\ln 2}{1590} \cdot t} = \frac{3}{10} \]

\[ -\frac{\ln 2}{1590} \cdot t = \ln \left( \frac{3}{10} \right) \]

\[ t = -\frac{\ln \left( \frac{3}{10} \right)}{\ln 2}. \quad 1590 \approx 2672 \text{ yrs} \]

\[ \text{Related Rates} \]

(sphere)

**Ex.** Air being pumped into a balloon such that the volume is increasing by 50 cm\(^3\)/s. How fast is the radius of the balloon increasing when \( r = 10 \text{ cm} \)?

Know \( \frac{dV}{dt} \) \( V \) = volume \quad Want \( \frac{dr}{dt} \) \( r \) = radius

Use the relation: \[ V = \frac{4}{3} \pi r^3 \]

Apply \( \frac{d}{dt} \) to both sides: \[ \frac{dV}{dt} = \frac{4}{3} \pi \cdot 3r^2 \cdot \frac{dr}{dt} \]
Now plug in what we know at particular moment:

\[
\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}
\]

\[
50 = 4\pi (10)^2 \frac{dr}{dt}
\]

\[
\frac{dr}{dt} = \frac{50}{4\pi \cdot 100} = \frac{1}{8\pi} \approx 0.040 \text{ cm/s}
\]

10-ft ladder leaning against wall

Bottom of ladder moves away from wall at 10 ft/s. How fast does top of ladder move down the wall when it is 2 ft from the ground?

\[
x \text{ in ft}
\]

\[
t \text{ in s}
\]

Know \(\frac{dx}{dt} = 10\) \text{ Want } \frac{dy}{dt} \text{.}

Use \(x^2 + y^2 = 10^2\).

Apply \(\frac{d}{dt}\) to both sides:

\[
\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(10^2)
\]

\[
2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0
\]

Plug in: \(\frac{dx}{dt} = 10\)

\[
20x + 2y \cdot \frac{dy}{dt} = 0
\]

\[
\frac{dy}{dt} = -\frac{20x}{2y} = -\frac{10x}{y}
\]

To find \(x\) and \(y\):

\[
x^2 + 2^2 = 10^2
\]

\[
x = \sqrt{96} = 4\sqrt{6}
\]

\[
y = 2
\]

\[
\frac{dy}{dt} = -\frac{10 \cdot 4\sqrt{6}}{2} = -20\sqrt{6} \text{ ft/s}
\]
\[ \begin{align*}
\text{Ex:} & \quad x &= x(t) \quad y = \sqrt{2x+1} \quad \text{If} \quad \frac{dx}{dt} = 3, \text{what is} \quad \frac{dy}{dt} \quad \text{when} \quad x = 4? \\\n& \text{(1) Directly:} \quad \frac{dy}{dt} = \frac{1}{2} \frac{1}{\sqrt{2x+1}} \cdot 2 \cdot \frac{dx}{dt} \\
& = \frac{1}{\sqrt{2x+1}} \cdot \frac{dx}{dt} \quad \text{Plug in:} \quad \frac{1}{\sqrt{2x+1}} \cdot 3 = \frac{1}{3} \cdot 3 = 1 \\
& \text{so} \quad \frac{dy}{dt} = 1 \\
& \text{(2) Write} \quad y^2 = 2x+1, \text{then} \quad 2y \cdot \frac{dy}{dt} = 2 \cdot 1 \cdot \frac{dx}{dt} \\
& = \frac{1}{y} \cdot \frac{dx}{dt} = \frac{1}{\sqrt{2x+1}} \cdot 3 = 1
\end{align*} \]

\[ \begin{align*}
\text{Ex: Gas in a box} & \quad \text{Boyle’s Law:} \quad PV = C \\
& \quad P = \text{pressure of gas} \quad V = \text{volume of gas} \\
& \quad \text{Say} \quad V = 400 \text{ cm}^3 \\
& \quad P = 80 \text{ kPa, decreasing at 10 kPa/min} \\
& \text{At what rate is} \quad V \text{ increasing?} \quad \text{Know} \quad \frac{dP}{dt} = -10 \quad \text{want} \quad \frac{dV}{dt} \\
& PV = C \\
& \frac{d}{dt}(PV) = \frac{d}{dt}(C) \\
& \frac{dP}{dt} \cdot V + P \cdot \frac{dV}{dt} = 0 \\
& P \frac{dV}{dt} = -V \frac{dP}{dt} \\
& \frac{dV}{dt} = -\frac{V}{P} \frac{dP}{dt} \quad \text{dP \quad dt = -10 \quad P = 80 \quad V = 400} 
\end{align*} \]
\[ \frac{dV}{dt} = \frac{-400}{80} \cdot (-10) = 50 \text{ cm}^3/\text{min} \]

**Ex.** A plane flies at altitude 5 km directly over a tracking station.

When angle of elev. \( \Theta = \frac{\pi}{3} \) rad
and \( \frac{d\Theta}{dt} = -\frac{\pi}{6} \text{ rad/min} \)

how fast is the plane moving?

\[ \tan \Theta = \frac{5}{x} \quad \rightarrow \quad \sqrt{3} = \frac{5}{x} \quad x = \frac{5}{\sqrt{3}} \]

\[ \sec^2 \Theta \frac{d\Theta}{dt} = -\frac{5}{x^2} \frac{dx}{dt} \]

plug in: \( \sec \Theta = 2, \quad x = \frac{5}{\sqrt{3}}, \quad \frac{d\Theta}{dt} = -\frac{\pi}{6} \)

so \[ 4 \left( -\frac{\pi}{6} \right) = -\frac{5}{\left( \frac{25}{3} \right)} \frac{dx}{dt} \]

\[ -\frac{2\pi}{3} = -\frac{15}{25} \frac{dx}{dt} \]

\[ \frac{2\pi}{3} = \frac{3}{5} \frac{dx}{dt} \]

\[ \frac{10\pi}{9} = \frac{dx}{dt} \]

\[ 3.49 \text{ km/min} \]