Last time: maxima and minima of functions

**Mean Value Theorem**

**Fact:** Suppose $f$ is a function continuous on $[a,b]$ and differentiable on $(a,b)$.

Then there is some $c$ in $(a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

**Example:** Let $f(x) = x^2 - x$

The slope of the tangent line at $(c, f(c))$ is equal to the slope of the secant line connecting $(a, f(a))$ and $(b, f(b))$.

**Example:** Let $f(x) = x^2 - x$

The slope of the secant line is $\frac{6 - 0}{2 - 0} = 3$

MVT says there must be some $c$ in $(0, 2)$ such that $f'(c) = 3$

Let's check: $f'(x) = 3x^2 - 1$

so $f'(c) = 3$ means $3c^2 - 1 = 3$

$3c^2 = 4$  \hspace{1cm} $c^2 = \frac{4}{3}$ \hspace{1cm} $c = \frac{2}{\sqrt{3}} > 1$

$\checkmark$ It looks fine!
Ex Suppose we drive 200 miles in 5 hours.
Let the position be \( x(t) \).
\[
\begin{align*}
  x(0) &= 0 \\
  x(5) &= 200
\end{align*}
\]
Slope of secant line = \( \frac{200}{5} = 40 \) miles/hour
\((= \text{average speed})\)

\[\text{MVT} \implies \] at some moment, the speedometer read exactly \( 40 \) mph.

\[\text{Graphing using derivatives}\]

How do we use \( f'(x) \) to get information about the graph of \( f(x) \)?

Ex Find where the function \( f(x) = 3x^4 - 4x^3 - 12x^2 + 5 \)
is increasing and where it is decreasing.

\[
f'(x) = 12x^3 - 12x^2 - 24x
\]
\[
= 12x(x^2 - x - 2)
\]
\[
= 12x(x + 1)(x - 2)
\]

To see whether \( f'(x) \) is +ve or -ve, look at these 3 pieces:

\[
\begin{array}{cccc}
\text{sign of } f'(x) & - & - & - \\
\text{at } x & -1 & 0 & +
\end{array}
\]

So \( f(x) \) is increasing for \( x \) in \((-1,0) \cup (2,\infty)\).

\( f(x) \) is decreasing for \( x \) in \((-\infty,-1) \cup (0,2)\).

\[
\begin{align*}
  f(-1) &= 0 \\
  f(0) &= 5 \\
  f(2) &= -27
\end{align*}
\]

Let's look closer at the critical points.
\( f'(x) = 0 \) at \( x = -1, 0, 2 \).
At $x = 2$:  
\[
\begin{array}{c|c|c}
\text{sign of } f'(x) & - & + \\
\hline
x & \hline
\end{array}
\]
local minimum

At $x = 0$:  
\[
\begin{array}{c|c|c}
\text{sign of } f'(x) & + & - \\
\hline
x & \hline
\end{array}
\]
local maximum

At $x = -1$:  
\[
\begin{array}{c|c|c}
\text{sign of } f'(x) & - & + \\
\hline
x & \hline
\end{array}
\]
local minimum

**First Derivative Test**

If $c$ is a critical number for $f(x)$,

1. If $f'(x)$ changes sign from the left at $c$, then $f$ has local max at $c$.

2. If $f'(x)$ changes sign from the right to the left at $c$, then $f$ has local min at $c$.

3. If $f'(x)$ does not change sign at $c$, then $f$ has neither max nor min at $c$.

**Example**

Find all local max/min of $f(x) = x^{\frac{1}{3}}(x+4)$ on $(0, \infty)$ and $(-\infty, 0)$

Find critical numbers:  
\[
f(x) = x^{\frac{1}{3}} + 4x^{\frac{1}{3}}
\]
\[
f'(x) = \frac{1}{3} x^{-\frac{2}{3}} + \frac{4}{3} x^{-\frac{1}{3}}
\]
\[
= \frac{1}{3} \left( x^{-\frac{2}{3}} + 4x^{-\frac{1}{3}} \right)
\]
\[
= \frac{1}{3} x^{-\frac{2}{3}} (x+1)
\]

$f'(x) = 0$ only at $x = -1$.

\[\rightarrow \text{ on } (0, \infty): \text{ no local max/min} \]
\[\rightarrow \text{ on } (-\infty, 0): \text{ local min} \]

So $x = -1$ is a local minimum.
Concavity

Both of these have $f'(x) > 0$ for all $x \in (a, b)$
but they are different:
Say graph of $y = f(x)$ is concave up on $(a, b)$ if it lies above all of its tangent lines in $(a, b)$.
Say graph of $y = f(x)$ is concave down on $(a, b)$ if it lies below all of its tangent lines in $(a, b)$.

Fact
If $f''(x) < 0$ for all $x \in (a, b)$ then the graph of $f$ is concave down on $(a, b)$.
If $f''(x) > 0$ for all $x \in (a, b)$ then the graph of $f$ is concave up on $(a, b)$.

Ex
$f(x) = x^2$
$f'(x) = 2x$
$f''(x) = 2$ so $f''(x) > 0$ for all $x \in (-\infty, \infty)$
so the graph $y = x^2$ is concave up for all $x \in (-\infty, \infty)$. 
A point of inflection is a point \((c, f(c))\) where \(f\) is continuous and the graph \(y = f(x)\) changes from concave up to concave down or vice versa.

**Ex** \(f(x) = x^3 - 3x\)

\[
f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x+1)(x-1)
\]

\[
f''(x) = 6x
\]

So: \(x = 1\) is local minimum
\(x = 0\) is inflection point
\(x = -1\) is local maximum

**Second Derivative Test**

If \(f\) is continuous at \(c\), \(f'(c) = 0\), and

1. \(f''(c) > 0\), then \(c\) is local minimum
2. \(f''(c) < 0\), then \(c\) is local maximum
3. \(f''(c) = 0\), then the test fails — gives no information

**Ex** Sketch \(y = x^4 - 4x^3\).

\[
f(x) = x^4 - 4x^3
\]

\[
f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)
\]

\[
f''(x) = 12x^2 - 24x = 12x(x-2)
\]

**Ex** Sketch \(y = x^4 - 4x^3\).

\[
f(x) = x^4 - 4x^3
\]

\[
f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)
\]

\[
f''(x) = 12x^2 - 24x = 12x(x-2)
\]
critical pts:  \( x = 0 \)  neither max nor min  
\( x = 3 \)  local min  

inflection pts:  \( x = 0, x = 2 \)

\[
\begin{align*}
f(0) &= 0 \\
f(2) &= 16 - 32 = -16 \\
f(3) &= 81 - 108 = -27
\end{align*}
\]