Lecture 15 | 5 Oct 2018

Exponential growth and exponential decay

Suppose we have a function $P(t)$ such that $\frac{dP}{dt}$ is proportional to P itself.

time

rate of change of P

ie $\frac{dP}{dt} = k \cdot P$ for some constant k.

Then what is $P(t)$?

$P(t) = t^2$?
$\frac{dP}{dt} = 2t \neq k \cdot t^2 = kP$

$P(t) = e^t$?
$\frac{dP}{dt} = e^t \neq k \cdot e^t = k \cdot P$

$P(t) = e^{kt}$?
$\frac{dP}{dt} = k \cdot e^{kt} = k \cdot e^{kt} = k \cdot P$ ✓

One possibility: $P(t) = e^{kt}$

[then $\frac{dP}{dt} = ke^{kt} = k \cdot P$ ✓]

Another poss: $P(t) = 0$

Another: $P(t) = Ce^{kt}$

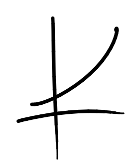
[then $\frac{dP}{dt} = C \cdot k \cdot e^{kt} = k \cdot P$ ✓]

This is actually the most general solution!

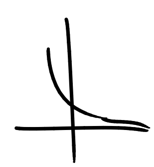
So: $\frac{dP}{dt} = k \cdot P(t) \Longrightarrow P(t) = C \cdot e^{kt}$ for some C.

Examples:

1. population growth under consistent conditions $(k > 0)$



2. radioactive decay $(k < 0)$



3. compound interest $(k > 0)$

Ex A population of bacteria grows from 1g at 2pm to 15g at 5pm.

What will be the mass of bacteria at 10pm?

$$P(t) = Ce^{kt}$$

(mass of bac. at time t in grams)

(t = # hours past 2pm)

$$P(0) = 1 \Rightarrow C \cdot e^{k \cdot 0} = 1$$

ie $C = 1$

$$P(3) = 15 \Rightarrow C \cdot e^{k \cdot 3} = 15$$

$$e^{3k} = 15$$

$$3k = \ln 15$$

$$k = \frac{1}{3} \ln 15$$

so $P(t) = 1 \cdot e^{\left(\frac{1}{3}\ln 15\right)t}$

We want $P(8)$. $P(8) = e^{\left(\frac{1}{3}\ln 15\right) \cdot 8} = e^{\frac{8}{3}\ln 15} \approx 1368\text{ g bacteria}$

Q: Why do we have to use e instead of some other base?

A: Actually can use any base:

$$2^{kt} = \left(e^{\ln 2}\right)^{kt} = e^{(k \ln 2)t} = e^{k't}$$

$$k' = k \ln 2$$

Ex The $\frac{1}{2}$-life of radium-226 is 1590 yrs.
Suppose we have 100 mg of radium-226.
When will it be reduced to 30 mg?

$$P(t) = Ce^{kt}$$

P in mg
t in years

$$P(0) = 100 \Rightarrow Ce^{k \cdot 0} = 100$$

$$C = 100$$

$$\ln\left(\frac{1}{x}\right) = -\ln x$$
$$\ln\left(x^{-1}\right)$$

"$\frac{1}{2}$-life" means the amount of time it takes $P(t)$ to be reduced by $\frac{1}{2}$.

$\frac{1}{2}$-life is 1590 yrs $\Rightarrow$ $e^{k \cdot (1590)} = \frac{1}{2}$

$$k \cdot 1590 = \ln\left(\frac{1}{2}\right) \qquad k = \frac{\ln\left(\frac{1}{2}\right)}{1590} = -\frac{\ln 2}{1590}$$

Want to find t such that $P(t) = 30$

$$C \cdot e^{kt} = 30$$
$$100 \cdot e^{\left(-\frac{\ln 2}{1590}\right)t} = 30$$
$$e^{-\frac{\ln 2}{1590}t} = \frac{3}{10}$$
$$-\frac{\ln 2}{1590}t = \ln\left(\frac{3}{10}\right)$$
$$t = -\ln\left(\frac{3}{10}\right) \cdot \frac{1590}{\ln(2)} \approx \underline{2672 \text{ yrs}}$$

Related Rates

Ex Air is being pumped into a spherical balloon such that the volume is increasing by $50\ \text{cm}^3/\text{s}$.

How fast is the radius of the balloon increasing, when $r = 10$ cm?

Know $\frac{dV}{dt}$ V = volume

Want $\frac{dr}{dt}$ r = radius

Use the relation: $V = \frac{4}{3}\pi r^3$

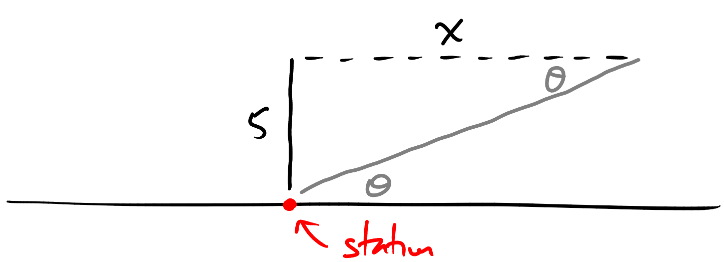
Apply $\frac{d}{dt}$ to both sides: $\frac{dV}{dt} = \frac{4}{3}\pi 3r^2 \frac{dr}{dt}$

(t in sec)

(r in cm) (V in cm^3)

$$50 = 4\pi (10)^2 \frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{50}{400\pi} = \frac{1}{8\pi} \approx \underline{0.04\ \text{cm/s}}$$

Ex A plane flies at altitude 5km directly over a tracking station.



When angle of elev. $\theta = \frac{\pi}{3}$ rad

and $\frac{d\theta}{dt} = -\frac{\pi}{6}$ rad/min

how fast is the plane moving?

Speed of plane = $\frac{dx}{dt}$ We know: θ, $\frac{d\theta}{dt}$

Relation: $\tan\theta = \frac{5}{x}$ take $\frac{d}{dt}$ both sides:

$$\sec^2\theta \frac{d\theta}{dt} = -\frac{5}{x^2}\frac{dx}{dt}$$

plug in: $\theta = \frac{\pi}{3} \longrightarrow \sec\theta = 2$, $\frac{d\theta}{dt} = -\frac{\pi}{6}$, $x = \frac{5}{\tan\frac{\pi}{3}} = 5/\sqrt{3}$

so $$4\cdot\left(-\frac{\pi}{6}\right) = -\frac{5}{(5/\sqrt{3})^2}\frac{dx}{dt}$$

solve: $$\frac{10\pi}{9} = \frac{dx}{dt}$$

$\approx$

$\underline{3.49 \text{ km/min}}$