Lecture 16

8 Oct 2018

Last time: related rates

Ex Gas in a box.

Boyle's Law: $PV = C$

P = pressure of gas
V = volume of gas

Say $V = 400 \text{ cm}^3$
$P = 80$ kPa, decreasing at 10 kPa/min

At what rate is V increasing?

Know $P, V, \frac{dP}{dt}$ Want $\frac{dV}{dt}$

$$P(t)V(t) = C$$

$$PV = C$$

$$\frac{d}{dt}(PV) = \frac{d}{dt}(C)$$

$$\frac{dP}{dt}\cdot V + P\cdot\frac{dV}{dt} = 0$$

$$P\frac{dV}{dt} = -V\frac{dP}{dt}$$

$$\frac{dV}{dt} = -\frac{V}{P}\frac{dP}{dt}$$

$P = 80$ $\frac{dP}{dt} = -10$
$V = 400$

P in kPa
V in cm^3
t in min

$$= -\frac{400}{80}\cdot(-10) = -5\cdot(-10) = 50$$

$50 \frac{\text{cm}^3}{\text{min}}$

Ex $x = x(t)$ $y = \sqrt{2x+1}$ If $\frac{dx}{dt} = 3$, what is $\frac{dy}{dt}$?
$y = y(t)$ $x = 4$,

Two approaches:

① direct — $\frac{dy}{dt} = \frac{d}{dt}\sqrt{2x+1} = \frac{d}{dt}(2x+1)^{1/2}$

$$= \frac{1}{2}(2x+1)^{-1/2}\frac{d}{dt}(2x+1)$$

$$= \frac{1}{2}(2x+1)^{-1/2}\left(2\frac{dx}{dt}\right)$$
$$= \frac{1}{2}(2\cdot 4+1)^{-1/2}(2\cdot 3)$$
$$= \frac{1}{2}(9^{-1/2})\cdot 6 = \frac{1}{2}$$

② square both sides

$y = \sqrt{2x+1}$

$\hookrightarrow\ y^2 = 2x+1$

$2y\frac{dy}{dt} = 2\frac{dx}{dt}$

$\frac{dy}{dt} = \frac{1}{y}\frac{dx}{dt} = \frac{1}{3}\cdot 3 = \frac{1}{2}$

$x = 4 \longrightarrow y = \sqrt{9} = 3$

$\frac{dx}{dt} = 3$

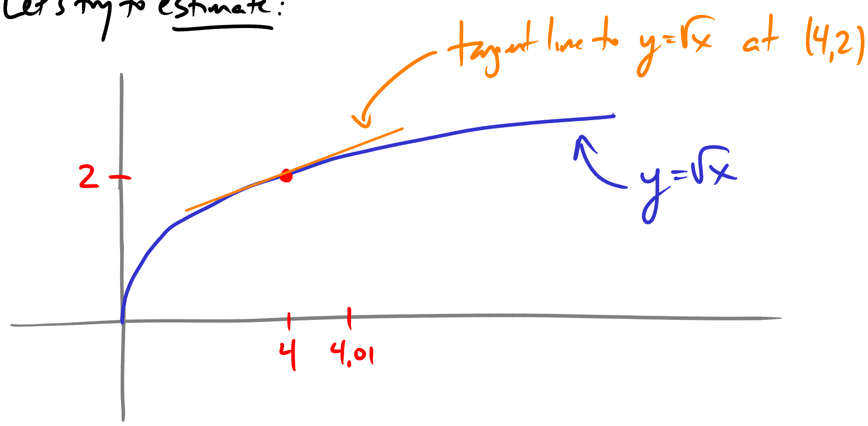
Another use of derivatives.

Linear Approximation

What is $\sqrt{4}$? 2

What is $\sqrt{4.01}$? "A little bigger than 2"

Let's try to estimate:



Tangent line: $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$.

At $x=4$: $\frac{dy}{dx} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$.

So, tangent line is line thru $(4,2)$ with slope $\frac{1}{4}$:

$y - 2 = \frac{1}{4}(x-4)$

ie $y = 2 + \frac{1}{4}(x-4)$

Plug in $x = 4.01$: $y = 2 + \frac{1}{4}(4.01-4)$

$= 2 + \frac{1}{4}(0.01) = \underline{2.0025}$

So 2.0025 is our estimate of $\sqrt{4.01}$. The exact value $\sqrt{4.01} = 2.002498\ldots$

Similarly: estimate $\sqrt{4.03}$. Tangent line at $(2,4)$ is $y = 2 + \frac{1}{4}(x-4)$

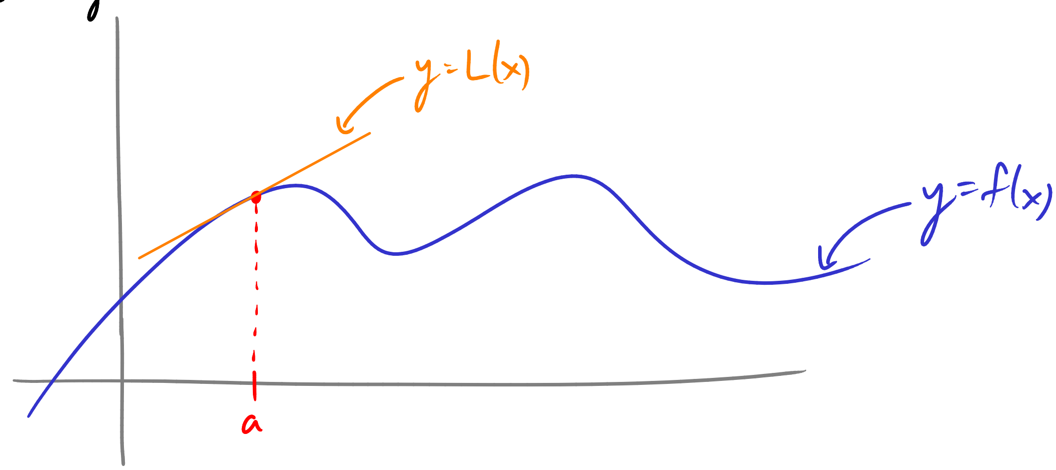
plug in $x = 4.03$: $\quad y = 2 + \frac{1}{4}(4.03-4)$

$$= 2 + \frac{1}{4}(0.03) = \underline{2.0075}$$

So 2.0075 is our estimate of $\sqrt{4.03}$. Exact value $\sqrt{4.03} = 2.007489\ldots$

x	$2+\frac{1}{4}(x-4)$	$\sqrt{x}$
3.99	1.9975	1.997...
4	2	2
4.01	2.0025	2.002498...
4.03	2.0075	2.007498...
6	2.5	2.4495...
8	3	2.8284...

Generally:



$y = L(x)$ is the tangent line to $y = f(x)$ at $(a, f(a))$.

ie $y = L(x)$ is the line thru $(a, f(a))$ with slope $f'(a) = m$

ie
$$y - f(a) = m(x-a)$$
$$L(x) - f(a) = f'(a)(x-a)$$
$$L(x) = f(a) + f'(a)(x-a)$$

Ex Estimate $\cos\left(\frac{\pi}{2} + 0.02\right)$.

Write $f(x) = \cos x$. Let $a = \frac{\pi}{2}$.

Linear approx to $f(x)$ at $a = \frac{\pi}{2}$:

$$f(a) = \cos\left(\frac{\pi}{2}\right) = 0.$$
$$f'(a) = -\sin\left(\frac{\pi}{2}\right) = -1.$$

$$L(x) = f(a) + f'(a)(x-a)$$
$$= 0 + (-1)\cdot\left(\frac{\pi}{2} + 0.02 - \frac{\pi}{2}\right) \qquad x = \frac{\pi}{2} + 0.02$$
$$= 0 + (-1)(0.02)$$

$$= -0.02.$$

So our estimate is $\cos\left(\frac{\pi}{2} + 0.02\right) \approx \underline{\underline{-0.02.}}$