Last time: optimization

**Example:** A cylindrical can without a top is to hold \( V \text{ cm}^3 \) of liquid.

What are the dimensions for the can which minimize the cost of metal for making the can?

\[
V = \pi r^2 h
\]

Want to minimize
\[
A = \pi r^2 + 2\pi rh
\]

Eliminate the variable \( h \):
\[
V = \pi r^2 h
\]
\[
\Rightarrow h = \frac{V}{\pi r^2}
\]

So
\[
A = \pi r^2 + 2\pi r \left( \frac{V}{\pi r^2} \right) = \pi r^2 + \frac{2V}{r}
\]

Want to find absolute minimum allowed value for \( A \): 

- \( A \) is a function of one variable \( r \) 
- \( A \) domain \( (0, \infty) \)

\[
A'(r) = 2\pi r - \frac{2V}{r^2}
\]

\[
= 2\pi r (1 - \frac{V}{\pi r^3})
\]

\[
A'(r) = 0 \text{ just at } 1 - \frac{V}{\pi r^3} = 0 \Rightarrow \frac{V}{\pi r^3} = 1 \Rightarrow \frac{V}{\pi} = r^3 \Rightarrow r = \sqrt[3]{\frac{V}{\pi}}
\]

\[\begin{array}{c|c|c}
 r & A(r) & A'(r) \\
\hline
 0 & \text{undefined} & \text{undefined} \\
 \sqrt[3]{\frac{V}{\pi}} & A(r) & \text{converges to zero}
\end{array}\]
So, the absolute minimum occurs at $r = \frac{3\sqrt{V}}{\pi}$

Then $h = \frac{V}{\pi r^2} = \frac{V}{\pi \left(\frac{V}{\pi}\right)^{2/3}} = \frac{V}{\pi} \frac{2/3}{\left(\frac{V}{\pi}\right)^{2/3}} = \frac{3\sqrt{V}}{\pi}$ also.

Ex. Find the largest possible area for a rectangle inscribed in a circle of radius 1.

\[ x^2 + y^2 = 1 \]
\[ A = (2x)(2y) = 4xy \]

One approach: eliminate a variable, say \( y \), by \( y = \sqrt{1-x^2} \)

then write \[ A = 4xy = 4x\sqrt{1-x^2} \]

now \( A \) is a function of one variable \( x \), Domain: \( x \in [0,1] \).

To find max for \( A \): ① find critical pts

\[
A'(x) = 4\sqrt{1-x^2} + 4x \left( \frac{-2x}{2\sqrt{1-x^2}} \right)
\]
\[
= 4\left( \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} \right)
\]
\[
= 4\left( \frac{1-x^2}{\sqrt{1-x^2}} - \frac{x^2}{\sqrt{1-x^2}} \right)
\]
\[
= \frac{4}{\sqrt{1-x^2}} (1 - 2x^2)
\]

so \( A'(x) = 0 \) just if \( 2x^2 = 1 \), i.e. \( x^2 = \frac{1}{2} \), i.e. \( x = \frac{1}{\sqrt{2}} \).

Value at critical pt: \( A\left(\frac{1}{\sqrt{2}}\right) = 4\left(\frac{1}{\sqrt{2}}\right)\sqrt{1 - \left(\frac{1}{\sqrt{2}}\right)^2} = 4\left(\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} = 2 \).
(2) values at endpoints. \[ A(0) = 0 \sqrt{1} = 0 \]
\[ A(1) = 1 \sqrt{0} = 0 \]

(3) max occurs at \( A\left(\frac{1}{2}\right) = 2 \), i.e. at \( x = \frac{1}{\sqrt{2}} \). Then \( y = \sqrt{1-x^2} = \sqrt{1-\frac{1}{2}} = \frac{1}{\sqrt{2}} \).

\[ \rightarrow \text{best possible area obtained by taking a square with side length } 2x = \frac{2}{\sqrt{2}} = \sqrt{2}. \]

Second approach: \[ A = 4xy \]

Differentiate: \[ \frac{dA}{dx} = 4y + 4x \frac{dy}{dx} \]
\[ 2x + 2y \frac{dy}{dx} = 0 \]
\[ \frac{dy}{dx} = -\frac{x}{y} \]

So \[ \frac{dA}{dx} = 4y + 4x \left(-\frac{x}{y}\right) = 4y - 4 \frac{x^2}{y} \]
\[ \frac{dA}{dx} = 0 \text{ if } 4y = 4 \frac{x^2}{y} \]
\[ y^2 = x^2 \]
\[ y = x \] (both \( x, y > 0 \))

and \( x^2 + y^2 = 1 \), so \( 2x = 1 \), i.e. \( x = \frac{1}{\sqrt{2}} \) — (faster way of getting the critical point)

**Antiderivatives**

- \( f(x) = x^2 \implies f'(x) = 2x \)
- \( f(x) = \sin(x^3) \implies f'(x) = 3x^2 \cos(x^3) \)
Suppose we want to “go backwards”:

\[ f \xrightarrow{\text{derivative}} f' \]

\[ f \xrightarrow{\text{antiderivative}} F \]

We say \( F(x) \) is an antiderivative for \( f(x) \) if \( F'(x) = f(x) \).

Ex. \( f(x) = x \) has antiderivative \( \frac{1}{2}x^2 \)

\[ = \frac{1}{2}x^2 + 2 \]

\[ = \frac{1}{2}x^2 + 37π \]

To get all possible antiderivatives of \( f(x) \), first find one antiderivative, and then add an arbitrary constant. (usually called \( C \))

Ex. \( f(x) = \cos x \) has general antiderivative \( F(x) = \sin x + C \)

\( f(x) = x^n \) has general antiderivative \( F(x) = \frac{x^{n+1}}{n+1} + C \) \( \left( \frac{d}{dx} F(x) = \frac{1}{n+1}(n+1)x^n = x^n \right) \)

\( f(x) = \frac{1}{1+x^2} \) has general antiderivative \( F(x) = \tan^{-1} x + C \)

\( f(x) = \frac{1}{x} \) has general antiderivative \( F(x) = \ln x + C \)

Build more complicated examples from there:

\( f(x) = 9x^2 + 6x^{3/2} - \frac{2}{x^4} + \cos 2x \)

has general antideriv.

\( F(x) = 3x^3 + 6 \left( \frac{1}{1/3} x^{5/3} \right) - 2 \left( \frac{1}{-3} x^{-3} \right) + \frac{1}{2} \sin 2x + C \)
Sometimes we don’t want the most general antiderivative, we want some specific one.

Ex. What is the function $F(x)$ which has $F'(x) = 4x + 7$ and $F(1) = 6$?

Since $F'(x) = 4x + 7$, we have $F(x) = 2x^2 + 7x + C$;

and $F(1) = 6$, so

$$2(1^2) + 7(1) + C = 6$$
$$9 + C = 6$$
$$C = -3$$

Thus $F(x) = 2x^2 + 7x - 3$

Why care about antiderivatives?

Standard reason:

- derivative of position is velocity
- derivative of velocity is acceleration
- antiderivative of acceleration is velocity
- antiderivative of velocity is position

Ex. A train accelerates with constant acceleration $a(t) = 4 \text{ ft/s}^2$.

At time $t = 0$ it has velocity $100 \text{ ft/s}$.

How far does it go in 20 s?

\[ a(t) = 4 \]
\[ v(0) = 100 \]
\[ s(0) = 0 \]

We want $s(20)$.

$s$ is antideriv. of $v$
$v$ is antideriv. of $a$
$v(t) = 4t + C$
and \( v(0) = 100 \), so \( 4 \cdot (0) + C = 100 \), i.e. \( C = 100 \).

So \( v(t) = 4t + 100 \).

Then \( s(t) = 2t^2 + 100t + D \)

and \( s(0) = 0 \), so \( 0 + 0 + D = 0 \), i.e. \( D = 0 \).

So \( s(t) = 2t^2 + 100t \).

\[
s(20) = 2(20^2) + 100(20) = 800 + 2000 = 2800 \text{ ft}.
\]

\[
\lim_{x \to \infty} \frac{x}{3} \cdot \ln \left( \frac{x + b}{x} \right)
\]

looks like \( \infty \cdot 0 \)

\[
= \lim_{x \to \infty} \frac{\ln \left(1 + \frac{b}{x}\right)}{\left(\frac{3}{x}\right)}
\]

\[
= \lim_{x \to \infty} \frac{\frac{1}{1 + b/x} \cdot \left(-\frac{b}{x^2}\right)}{-\frac{3}{x^2}} = 2 \lim_{x \to \infty} \frac{1}{1 + \frac{b}{x}} = \frac{2}{2} = 1
\]