Ex: A cylindrical can without a top is to hold \( V \) cm\(^3\) of liquid. What are the dimensions for the can which minimize the cost of metal?

Want to minimize surface area: \( A = \pi r^2 + 2\pi rh \)

Eliminate \( h \) using our constraint:
\[
V = \pi r^2 h \\
\frac{V}{\pi r^2} = h
\]

Then \( A = \pi r^2 + 2\pi r \left( \frac{V}{\pi r^2} \right) = \pi r^2 + \frac{2V}{r} \)

function of one variable \( r \), domain \( (0, \infty) \).

To find absolute minimum: look at \( A'(r) \)

\[
A'(r) = 2\pi r - 2\frac{V}{r^2}
\]

\[
A'(r) = 2\pi r \left( 1 - \frac{V}{\pi r^3} \right)
\]

\( A'(r) = 0 \) just if \( 1 - \frac{V}{\pi r^3} = 0 \)
\[
\frac{V}{\pi} = r^2 \quad 3\sqrt{\frac{V}{\pi}} = r
\]

Thus the absolute minimum of \( A(r) \) is attained at \( r = \frac{3\sqrt{V}}{\sqrt{\pi}} \).

\[
h = \frac{V}{\pi r^2} = \frac{\sqrt{V}}{\pi \left(\frac{3\sqrt{V}}{\sqrt{\pi}}\right)^2} = \frac{V}{\pi} \left(\frac{\sqrt{V}}{\pi}\right)^{\frac{3}{2}} = \left(\frac{V}{\pi}\right)^{\frac{3}{2}} \sqrt{\frac{V}{\pi}}
\]

**Example:** Find the largest area possible for a rectangle inscribed in a circle of radius 1.

\[x^2 + y^2 = 1\]

\[A = (2x)(2y) = 4xy\]

First approach: eliminate \( y \)

\[y = \sqrt{1-x^2}\]

Then \( A = 4x\sqrt{1-x^2} \), a function of a single variable, \( A(x) \), with domain \([0, 1]\).

1. Find critical points: \( A'(x) = 4\left(\sqrt{1-x^2} + x \frac{d}{dx} \sqrt{1-x^2}\right)

   \[= 4\left(\sqrt{1-x^2} + x \frac{-2x}{2\sqrt{1-x^2}}\right)\]

   \[= 4\left(\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}\right)\]

   \[= 4\left(\frac{1-x^2}{\sqrt{1-x^2}} - \frac{x^2}{\sqrt{1-x^2}}\right)\]

   \[= 4\left(\frac{1-2x^2}{\sqrt{1-x^2}}\right)\]
\[
= 4 \left( \frac{1-x^2}{\sqrt{1-x^2}} - \frac{x^2}{\sqrt{1-x^2}} \right) \\
= \frac{4}{\sqrt{1-x^2}} \left( 1-2x^2 \right)
\]

so \( A'(x) = 0 \) just if \( 1 - 2x^2 = 0 \) ie \( 2x^2 = 1 \)

\[ x^2 = \frac{1}{2} \]

\[ x = \frac{1}{\sqrt{2}} \] (not \( -\frac{1}{\sqrt{2}} \), this isn't in domain)

So \( x = \frac{1}{\sqrt{2}} \) is the only critical pt.

\[
A\left( \frac{1}{\sqrt{2}} \right) = 4 \left( \frac{1}{\sqrt{2}} \right) \cdot \sqrt{1 - \left( \frac{1}{\sqrt{2}} \right)^2}
\]

\[
= 4 \left( \frac{1}{\sqrt{2}} \right) \cdot \frac{\sqrt{2}}{2} = 2.
\]

Here \( y = \sqrt{1-x^2} = \sqrt{1-\left( \frac{1}{\sqrt{2}} \right)^2} = \frac{1}{\sqrt{2}} = x \)

so this rectangle is a square, with side length \( \frac{1}{\sqrt{2}} \).

(2) Check endpoints: \( A(0) = 4 \cdot 0 \cdot \sqrt{1} = 0 \)

\( A(1) = 4 \cdot 1 \cdot \sqrt{0} = 0 \).

So maximum area is 2, obtained at \( x = \frac{1}{\sqrt{2}} \), \( y = \frac{1}{\sqrt{2}} \).

Alternate way:

\[ A = 4xy \]

want to find places where \( \frac{dA}{dx} = 0 \).

Differentiate both equations:

\[
\frac{dA}{dx} = 4y + 4x \frac{dy}{dx}
\]

\[
2x + 2y \frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx} = -\frac{x}{y}
\]
\[ \frac{dA}{dx} = 4y - 4 \frac{x^2}{y} \]

So, \( \frac{dA}{dx} = 0 \) means

\[ 0 = 4y - 4 \frac{x^2}{y} \]

\[ 4y = 4 \frac{x^2}{y} \]

\[ y^2 = x^2 \]

\[ y = x \quad (x, y > 0) \]

So the critical pt. is a rectangle which is a square.

\[ x = y \]

\[ x^2 + y^2 = 1 \]

\[ 2x^2 = 1 \]

\[ x = \frac{1}{\sqrt{2}} \]

\[ A = 4 \sin \theta \cos \theta \]

\[ \frac{dA}{d\theta} = \cdots \]

or:

\[ A = 2 \sin 2\theta \]

\[ \frac{dA}{d\theta} = 4 \cos 2\theta \]

So, \( \frac{dA}{d\theta} = 0 \) at \( \cos 2\theta = 0 \)

\[ \theta = \frac{\pi}{4} \]

**Antiderivatives**

\[ f(x) = x^2 \implies f'(x) = 2x \]

\[ f(x) = \sin(x^3) \implies f'(x) = 3x^2 \cos(x^3) \]
Suppose we want to "go backwards":

\[ f' \rightarrow f \]  

\[ \text{antiderivative} \leftrightarrow \text{derivative} \]

Any function \( f \) has many antiderivatives!

**Ex:** \( f(x) = x \) has antiderivatives \( F(x) = \frac{1}{2}x^2 \)

\[
F(x) = \frac{1}{2}x^2 + 12 \\
F(x) = \frac{1}{2}x^2 + 7\pi - 8
\]

To get all possible antiderivs for \( f(x) \),
first find one antideriv, then add a arbitrary constant (usually called \( C \)).

**Ex:** \( f(x) = \cos x \) has general antiderivative \( F(x) = \sin x + C \)

\[
F(x) = \frac{x^{n+1}}{n+1} + C \quad \text{if } n \neq -1
\]

(e.g. \( f(x) = x^4 \) has antideriv. \( F(x) = \frac{x^5}{5} + C \))

\[
\begin{align*}
\text{\( f(x) = 9x^2 + 6x^{3/2} - \frac{2}{x^4} + \cos 2x \)} & \\
\text{has general antideriv.} & \\
F(x) &= 9 \cdot \frac{x^3}{3} + 6 \cdot \frac{x^{5/2}}{(5/2)} - 2 \cdot \frac{x^{-3}}{-3} + \frac{1}{2} \sin 2x + C \\
&= 3x^3 + 12x^{5/2} + \frac{2}{3}x^{-3} + \frac{1}{2} \sin 2x + C \\
\end{align*}
\]

\[
\begin{align*}
\text{\( f(x) = x^{-1} = \frac{1}{x} \) has general antideriv.} & \ln x + C \\
\end{align*}
\]

Sometimes we want a specific antideriv:

**Ex** What is the function \( F(x) \) such that \( F'(x) = 4x + 7 \) ?

and \( F(1) = 6 \)
Since $F'(x) = 4x + 7$

have $F(x) = 4\left(\frac{x^2}{2}\right) + 7(x) + C$

$= 2x^2 + 7x + C$

and $F(1) = 6$, so $2(1^2) + 7(1) + C = 6$

$9 + C = 6$

$C = -3$

so $F(x) = 2x^2 + 7x - 3$

**Why care about antideriv?**

One reason:

![Derivative and Antiderivative Diagram]

**Ex** A train accelerates with constant accel. $a(t) = 4 \text{ ft/s}^2$.

At time $t=0$ it has velocity $v(t=0) = 100 \text{ ft/s}$

and position $s(t=0) = 0 \text{ ft}$.

How far does it go in 20s? $s(t=20) = ?$

$a(t) = 4$

$v(t) = 4t + C$

$v(t=0) = 100$

$C = 100$

so $v(t) = 4t + 100$

$s(t) = 2t^2 + 100t + D$

$s(t=0) = 0$

$D = 0$

so $s(t) = 2t^2 + 100t$
\[ s(20) = 2(20^2) + 100 \cdot 20 = 2800 \text{ ft} \]

\[ f(x) \quad \text{A critical point of } f \text{ is a point } x \text{ s.t. } x \text{ is in domain of } f \quad \text{and} \quad \text{either} \quad f'(x) = 0 \quad \text{or} \quad f'(x) \text{ DNE} \]