Exam 2 average = 87%
Office hr today 4-5:30 RLM 9.13

HW due Fri 3am

Last time: antiderivatives

Ex. \( f(x) = x^4 + x \) has antiderivative \( \frac{x^5}{5} + \frac{x^2}{2} + C \) for any constant \( C \).

Res. Every continuous function \( f(x) \) has an antiderivative. But e.g. the antiderivative of \( f(x) = e^{-x^2} \) cannot be written in terms of "elementary" functions (+, -, ÷, exp, log, sin, cos, ...)

So we give it a new name: "error function"

Areas (Ch 5.1)

We all know the areas of simple shapes

\[
\int_{a}^{b} f(x) \, dx \quad A = \text{Area}
\]

How about more complicated shapes?

Could try to estimate area by counting boxes.
Not exact, but if grid squares very small, it's close.
(Get exact answer by taking a limit as the size of squares \( \rightarrow 0 \).)
Ex. Say \( f(x) = x^2 \).

Estimate the area of the region between \( y = f(x) \) and the x-axis, and between 0 and 1.

This is an overestimate of the actual area.

It is the "estimated area using 4 rectangles and using right endpoints as our sample points." So call it \( R_4 \): \( R_4 = \frac{15}{32} \).

Ex. Estimate area between graph of \( y = x^2 \) and x-axis, and between \( x = 0 \) and \( x = 1 \), using 5 rectangles and using left endpoints as sample points.
Estimated area \[ L_5 = \frac{1}{5} \cdot \left( 0^3 + \left( \frac{1}{5} \right)^3 + \left( \frac{2}{5} \right)^3 + \left( \frac{3}{5} \right)^3 + \left( \frac{4}{5} \right)^3 \right) \]

This is an underestimate of the actual area.

Now suppose we used 100 rectangles. Then we would get
\[ L_{100} = \frac{1}{100} \cdot \left( 0^3 + \left( \frac{1}{100} \right)^3 + \left( \frac{2}{100} \right)^3 + \cdots + \left( \frac{99}{100} \right)^3 \right) = 0.3283500 \]
\[ R_{100} = \frac{1}{100} \cdot \left( \left( \frac{1}{100} \right)^3 + \left( \frac{2}{100} \right)^3 + \left( \frac{3}{100} \right)^3 + \cdots + \left( \frac{99}{100} \right)^3 \right) = 0.3383500 \]

\[
\begin{array}{c|c|c}
 n & L_n & R_n \\
---&---&---
 10 & .285 & .385 \\
 100 & .3285 & .3385 \\
 1000 & .33285 & .33385 \\
\end{array}
\]

As \( n \to \infty \), both \( L_n \) and \( R_n \) approach \( \frac{1}{3} \). (e.g. \( R_n = \frac{(n+1)(2n+1)}{6n^2} \))

So, \( \frac{1}{3} \) is the exact area under the graph \( y = x^2 \) between \( x = 0 \) and \( x = 1 \).

For a general \( f(x) \), estimate the area similarly:

Width of each rectangle: \( \Delta x = \frac{b-a}{n} \)

Heights of rectangles: (using right endpoints)
\[ f(x_1), f(x_2), f(x_3), \ldots, f(x_n) \]
where \( x_i = x_0 + i \Delta x \)

\[ = a + i \Delta x \]

\[ \Rightarrow \text{estimated area} \quad R_n = \Delta x \left( f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n) \right) \]
Another convenient notation: ("sigma notation")

The symbol \( \sum_{i=1}^{n} f(x_i) \) means \( f(x_1) + f(x_2) + \ldots + f(x_n) \).

**Ex.** What is \( \sum_{i=1}^{4} i^2 \)?

\[
\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30.
\]

**Ex.** Write \( \frac{2^3}{n} + \frac{4^3}{n} + \frac{6^3}{n} + \ldots + \frac{(2n)^3}{n} \) in sigma notation.

\[
\frac{2^3}{n} + \frac{4^3}{n} + \ldots + \frac{(2n)^3}{n} = \sum_{i=1}^{n} \left( \frac{2i}{n} \right)^3.
\]

In this notation, \( R_n = \Delta x \sum_{i=1}^{n} f(x_i) \)

and similarly, \( L_n = \Delta x \sum_{i=1}^{n} f(x_{i-1}) \)

The actual area is \( A = \lim_{n \to \infty} R_n \) or \( A = \lim_{n \to \infty} L_n \) (both are the same).

**Ex.** Let \( A \) be the area of the region under the graph of \( f(x) = \sin^2 x \)

between \( x = \frac{\pi}{4} \) and \( x = \frac{3\pi}{4} \). Using right endpoints as sample points,

- Write a formula for \( A \) as a limit.

  \[
  a = \frac{\pi}{4}, \quad b = \frac{3\pi}{4}, \quad \Delta x = \frac{b-a}{n} = \frac{\frac{3\pi}{4} - \frac{\pi}{4}}{n} = \frac{\pi}{2n}
  \]

  \[
  x_i = a + i\Delta x = \frac{\pi}{4} + i\cdot\frac{\pi}{2n}
  \]

  \[
  R_n = \Delta x \sum_{i=1}^{n} f(x_i) = \frac{\pi}{2n} \cdot \sum_{i=1}^{n} \sin^2(x_i)
  \]

  \[
  = \frac{\pi}{2n} \cdot \sum_{i=1}^{n} \sin^2 \left( \frac{\pi}{4} + i\cdot\frac{\pi}{2n} \right)
  \]

  \[
  A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \left[ \frac{\pi}{2n} \sum_{i=1}^{n} \sin^2 \left( \frac{\pi}{4} + i\cdot\frac{\pi}{2n} \right) \right]
  \]
Estimate $A$ using 3 rectangles.

$$R_3 = \frac{\pi}{6} \sum_{i=1}^{3} \sin^2 \left( \frac{\pi}{4} + i \frac{\pi}{6} \right)$$

$$= \frac{\pi}{6} \left( \sin^2 \left( \frac{\pi}{4} + \frac{\pi}{6} \right) + \sin^2 \left( \frac{\pi}{4} + \frac{2\pi}{6} \right) + \sin^2 \left( \frac{\pi}{4} + \frac{3\pi}{6} \right) \right)$$

$$\approx 1.2388$$

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**Definite Integrals**

Say $f(x)$ is a function defined for $a \leq x \leq b$.

Divide $[a, b]$ into $n$ equal subintervals of width $\Delta x$, endpoints $x_0, x_1, \ldots, x_n$ where $x_0 = a, x_n = b$.

Pick any "sample points" $x_i^*$ in $[x_{i-1}, x_i]$.

The definite integral of $f$ from $a$ to $b$ is

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x$$

If that limit exists!

(It always does, if $f$ is continuous.)

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**Example**

Write the definition of $\int_1^3 \sqrt{x} \, dx$.

Divide the interval $[1, 3]$ into $n$ equal subintervals with width $\Delta x = \frac{3-1}{n} = \frac{2}{n}$.

Endpoints $x_0 = 1$, $x_1 = 1 + \Delta x$, $x_2 = 1 + 2\Delta x$, ...

$$x_i = 1 + i \Delta x = 1 + \frac{2i}{n}$$

Choose right endpoints: $x_i^* = x_i$

Then

$$\int_1^3 \sqrt{x} \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta x \cdot f(x_i^*) = \lim_{n \to \infty} \sum_{i=1}^{n} \left( \frac{2}{n} \right) \sqrt{1 + \frac{2i}{n}}$$