Exam 2 average ≈ 81%
My office hr today 4-5:30 RLM 9134
HW due Fri 3am

Last time: antiderivatives

\[ f(x) = x^4 + x \] has antiderivatives \[ \frac{x^5}{5} + \frac{x^2}{2} + C \]
Because \[ \frac{d}{dx} \left( \frac{x^5}{5} + \frac{x^2}{2} + C \right) = x^4 + x \].

Remark: Every continuous function has an antiderivative.
But, e.g., the antiderivative of \[ f(x) = e^{-x^2} \]
cannot be written in terms of "elementary" functions (+, −, ÷, exp, log, sin, cos, sin⁻¹, ...)

We give this antiderivative a new name: "error function"

Areas: We all know areas of simple shapes

How about more complicated shapes?

Let's try to estimate the area under the graph of \[ y = f(x) \]
over the x-axis between \( x = a \) and \( x = b \).
Ex: Say $f(x) = x^2$.

Estimate the area of the region between $y = f(x)$ and the $x$-axis, and between $x = 0$ and $x = 1$.

Idea: approximate our region by a bunch of rectangles.

\[
\begin{align*}
& A = \frac{1}{4} \left( \frac{1}{4} \right)^2 \\
& A = \frac{1}{4} \cdot \left( \frac{1}{2} \right)^2 \\
& A = \frac{1}{4} \cdot \left( \frac{3}{4} \right)^2 \\
& A = \frac{1}{4} \cdot \left( 1 \right)^2
\end{align*}
\]

Total area of rectangles = \( \frac{1}{4} \left[ \left( \frac{1}{4} \right)^2 + \left( \frac{1}{2} \right)^2 + \left( \frac{3}{4} \right)^2 + \left( 1 \right)^2 \right] \)

\[= \frac{15}{32}\]

This gives an overestimate of the area under $y = x^2$ from $x = 0$ to $x = 1$.

It is the "estimated area using 4 rectangles and using right endpoints of the intervals as sample points."

So call it $R_4$.

Thus $R_4 = \frac{15}{32}$. 

Ex. Estimate the same area using 5 rectangles and left endpoints as sample points.

Estimated area \[ L_5 = \frac{1}{5} \left( 0^2 + \left( \frac{1}{5} \right)^2 + \left( \frac{2}{5} \right)^2 + \left( \frac{3}{5} \right)^2 + \left( \frac{4}{5} \right)^2 \right) \]

\[ = \frac{30}{125} \]

This is an underestimate of the actual area.

Now say we use 100 rectangles. Then get \[ L_{100} = \frac{1}{100} \left( 0^2 + \left( \frac{1}{100} \right)^2 + \left( \frac{2}{100} \right)^2 + \cdots + \left( \frac{99}{100} \right)^2 \right) = 0.3285 \]

\[ R_{100} = \frac{1}{100} \left( \left( \frac{1}{100} \right)^2 + \left( \frac{2}{100} \right)^2 + \left( \frac{3}{100} \right)^2 + \cdots + \left( \frac{100}{100} \right)^2 \right) = 0.3385 \]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( L_n )</th>
<th>( R_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>.285</td>
<td>.385</td>
</tr>
<tr>
<td>100</td>
<td>.3285</td>
<td>.3385</td>
</tr>
<tr>
<td>1000</td>
<td>.33285</td>
<td>.33385</td>
</tr>
</tbody>
</table>

As \( n \to \infty \), both \( L_n \) and \( R_n \) approach \( \frac{1}{3} \).

So, \( \frac{1}{3} \) is the exact area between the graph \( y = x^2 \) and the \( x \)-axis and between \( x = 0 \) and \( x = 1 \).
For any continuous function \( f(x) \), estimate the area similarly: chop interval \([a,b]\) into \(n\) subintervals.

Width of each rectangle: \( \Delta x = \frac{b-a}{n} \)

Heights of rectangles: (using left endpoints) \( f(x_0), f(x_1), f(x_2), \ldots, f(x_{n-1}) \)

where \( x_0 = a \)
\[
x_1 = a + \Delta x \\
x_2 = a + 2\Delta x \\
\vdots
\]

\( \implies \) estimated area \( L_n = \Delta x \cdot (f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1})) \)

A convenient notation ("sigma notation"): the symbol \( \sum_{i=1}^{n} a_i \) means \( a_1 + a_2 + a_3 + \cdots + a_n \).

**Ex.** What is \( \sum_{i=1}^{4} i^2 \)?

\[
\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30.
\]

**Ex.** What is \( \sum_{i=1}^{6} 3i \)?

\[
3(1) + 3(2) + 3(3) + 3(4) + 3(5) + 3(6) \\
= 3 + 6 + 9 + 12 + 15 + 18 \\
= 63.
\]

**Ex.** Write \( \frac{1^3}{n} + \frac{2^3}{n} + \frac{3^3}{n} + \cdots + \frac{n^3}{n} \) in sigma notation.

\[
\sum_{i=1}^{n} \frac{i^3}{n}.
\]
In this notation, \[ L_n = \Delta x \sum_{i=1}^{n} f(x_{i-1}) \]
\[ R_n = \Delta x \sum_{i=1}^{n} f(x_i) \]

The actual area is \[ A = \lim_{n \to \infty} L_n \quad \text{or} \quad A = \lim_{n \to \infty} R_n \]  
(Both are the same!)

**Example**
Let \( A \) be the area of the region under the graph of \( f(x) = \sin^2 x \) between \( x = \frac{\pi}{4} \) and \( x = \frac{3\pi}{4} \). Using right endpoints as sample points,

\[ x_1, x_2, x_3, \ldots \]

\[ a = \frac{\pi}{4} \quad b = \frac{3\pi}{4} \]

- Write a formula for \( A \) as a limit,

\[ a = \frac{\pi}{4} \]
\[ b = \frac{3\pi}{4} \]
\[ \Delta x = \frac{b-a}{n} = \frac{\frac{3\pi}{4} - \frac{\pi}{4}}{n} = \frac{\pi}{2n} \]

\[ x_i = a + i\Delta x = \frac{\pi}{4} + i \cdot \frac{\pi}{2n} \]

\[ R_n = \Delta x \cdot \sum_{i=1}^{n} f(x_i) = \frac{\pi}{2n} \cdot \sum_{i=1}^{n} \sin^2 (x_i) \]

\[ = \frac{\pi}{2n} \cdot \sum_{i=1}^{n} \sin^2 \left( \frac{\pi}{4} + i \cdot \frac{\pi}{2n} \right) \]

The actual area is \[ A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \frac{\pi}{2n} \sum_{i=1}^{n} \sin^2 \left( \frac{\pi}{4} + i \cdot \frac{\pi}{2n} \right) \]

- Estimate \( A \) using 3 rectangles,

\[ R_3 = \frac{\pi}{6} \cdot \sum_{i=1}^{3} \sin^2 \left( \frac{\pi}{4} + i \cdot \frac{\pi}{6} \right) \]

\[ = \frac{\pi}{6} \left( \sin^2 \left( \frac{\pi}{4} + \frac{\pi}{6} \right) + \sin^2 \left( \frac{\pi}{4} + 2 \cdot \frac{\pi}{6} \right) + \sin^2 \left( \frac{\pi}{4} + 3 \cdot \frac{\pi}{6} \right) \right) \]

\[ \approx 1.23885 \]
Definite integrals

Say $f(x)$ is a function defined for $a \leq x \leq b$.

Divide $[a, b]$ into $n$ equal subintervals of width $\Delta x$, endpoints $x_0, x_1, x_2, \ldots, x_n$ with

$$x_i = a + i \Delta x$$

Pick any "sample points" $x_i^*$ in $[x_{i-1}, x_i]$.

The definite integral of $f$ from $a$ to $b$ is

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \left( \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x \right)$$

"Riemann sum"

if that limit exists!

(It always exists, if $f$ is continuous.)