Last time: definition of definite integral

Given a function $f$ on $[a,b]$
divide $[a,b]$ into $n$ equal subintervals, width $\Delta x = \frac{b-a}{n}$

$$x_i = x_0 + i\Delta x = a + i\frac{b-a}{n}$$

Pick "sample points" $x_i^*$ in the interval $[x_{i-1}, x_i]$

e.g. left endpoint: $x_i^* = x_{i-1}$
right endpoint: $x_i^* = x_i$

Then Riemann sum of $f$ is $\sum_{i=1}^{n} f(x_i^*) \Delta x$

and $\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \left( \sum_{i=1}^{n} f(x_i^*) \Delta x \right)$

**Ex** Write the definition of $\int_{-2}^{5} e^{x+1} \, dx$ as a limit, using right endpoints.

- $a = -2$
- $b = 5$
- $\Delta x = \frac{b-a}{n} = \frac{5-(-2)}{n} = \frac{7}{n}$
- $x_i^* = x_i = x_0 + i\Delta x = -2 + i\frac{7}{n}$

The function $f(x) = e^{x+1}$. 

Riemann sum: $\sum_{i=1}^{n} f(-2 + \frac{7i}{n}) \cdot \frac{7}{n}$

$$= \sum_{i=1}^{n} \left( e^{2(-2+\frac{7i}{n})+1} \right) \cdot \frac{7}{n}$$

$$= \frac{7}{n} \sum_{i=1}^{n} e^{-3+\frac{7i}{n}}$$

(Sum equality: $\sum_{i=1}^{n} c \cdot a_i = c \cdot \sum_{i=1}^{n} a_i$)
\[ S, \int_{-2}^{5} e^{2x+1} \, dx = \lim_{n \to \infty} \left( \frac{7}{n} \sum_{i=1}^{n} e^{-3 + \frac{14i}{n}} \right) \]

(If we used left endpoints, we’d get)
\[ \int_{-2}^{5} e^{2x+1} \, dx = \lim_{n \to \infty} \left( \frac{7}{n} \sum_{i=1}^{n} e^{-3 + \frac{M(i-1)}{n}} \right) \]

Both are OK!

**Facts about integrals**

\[ \int_{a}^{b} f(x) \, dx = A \]

\[ \int_{a}^{b} f(x) \, dx = A_1 - A_2 \]

\[ \int_{a}^{b} |f(x)| \, dx = A_1 + A_2 \]

**Ex** Evaluate \( \int_{1}^{3} x \, dx \) by interpreting it as an area.

\[ y = x \]

\[ A = \frac{1}{2} \cdot 2 \cdot 2 = 2 \]

\[ A = 2 \cdot 1 = 2 \]

\[ \text{Total area} = 4 \]

So, \( \int_{1}^{3} x \, dx = \frac{4}{2} \).
Ex. Evaluate \( \int_{0}^{1} -\sqrt{1-x^2} \, dx \) by interpreting it as an area.

\( y = -\sqrt{1-x^2} \) with \( x \) going from 0 to 1.
\( y^2 = 1-x^2 \), i.e. \( x^2 + y^2 = 1 \) (unit circle)

So \( \int_{0}^{1} -\sqrt{1-x^2} \, dx = \frac{\pi}{4} \)

This \( -\) sign because the graph of \( y = f(x) \) is below the \( x \)-axis!

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**Basic laws for integrals:**

1) \( \int_{a}^{b} c \, dx = c \cdot (b-a) \)

2) \( \int_{a}^{b} f(x) + g(x) \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx \)

3) \( \int_{a}^{b} cf(x) \, dx = c \int_{a}^{b} f(x) \, dx \)

4) \( \int_{a}^{b} f(x) - g(x) \, dx = \int_{a}^{b} f(x) \, dx - \int_{a}^{b} g(x) \, dx \)

5) \( \int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx \)

So for, always consider \( \int_{a}^{b} \) where \( b > a \).

Convenient to define \( \int_{b}^{a} \) \( f(x) \, dx = -\int_{a}^{b} f(x) \, dx \).

(e.g., we already saw \( \int_{1}^{3} x \, dx = 4 \). So, \( \int_{3}^{1} x \, dx = -4 \).)
\[ \text{and} \quad \int_{-1}^{1} \sqrt{1-x^2} \, dx = -\frac{\pi}{4}. \quad \text{So,} \quad \int_{-1}^{1} \sqrt{1-x^2} \, dx = \frac{\pi}{4}. \]

**Ex** If \( \int_{1}^{3} f(x) \, dx = 4 \) and \( \int_{3}^{7} f(x) \, dx = 16 \)

what is \( \int_{1}^{7} 3f(x) \, dx \)?

\[
\int_{1}^{7} f(x) \, dx = \int_{1}^{3} f(x) \, dx + \int_{3}^{7} f(x) \, dx = 4 + 16 = 20.
\]

So, \( \int_{1}^{7} 3f(x) \, dx = 3 \cdot \int_{1}^{7} f(x) \, dx = 3 \cdot 20 = 60. \]

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How do actually calculate definite integrals?

**Fundamental Theorem of Calculus**

**Fundamental Theorem of Calculus I:**

If \( F(x) = \int_{a}^{x} f(t) \, dt \) then \( F'(x) = f(x) \).

(i.e. \( \int_{a}^{x} f(t) \, dt \) is an antiderivative of \( f(x) \))

**Ex** What is the derivative of \( F(x) = \int_{-4}^{x} \sin t \, dt \)?

\[ F'(x) = \sin x \quad \text{by FTC I}. \]
Ex. What is the derivative of \( F(x) = \int_{4}^{x^2} \cos t \, dt \)?

Use chain rule:

\[
\frac{d}{dx} \int_{4}^{x^2} \cos t \, dt = \frac{du}{dx} \int_{4}^{u} \cos t \, dt = 2x \cdot \cos (x^2)
\]

Ex. \( \frac{d}{dx} \left( \int_{x}^{5} f(t) \, dt \right) = ? \)

\[
\int_{x}^{5} f(t) \, dt = -\int_{5}^{x} f(t) \, dt
\]

so \( \frac{d}{dx} \int_{x}^{5} f(t) \, dt = \frac{d}{dx} \left( -\int_{5}^{x} f(t) \, dt \right) = -\frac{d}{dx} \int_{5}^{x} f(t) \, dt = -f(x) \)

by FTC II

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**Fundamental Theorem of Calculus II:**

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a) \quad \text{where} \quad F(x) \text{ is any antiderivative of } f(x).
\]

(Exercise: Deduce FTC II from FTC I!)

**Notation:** \( F(b) - F(a) \) is sometimes written \( F \int_{a}^{b} \) or \( F \int_{a}^{b} \).

Ex. Calculate \( \int_{0}^{1} x^2 \, dx \).

Use FTC II: \( F(x) = \frac{1}{3} x^3 \) is an antiderivative of \( x^2 \), so

\[
F(1) - F(0) = \frac{1}{3} (1)^3 - \frac{1}{3} (0)^3 = \frac{1}{3}
\]
\[
\int_0^1 x^2 \, dx = \frac{1}{3} x^3 \bigg|_0^1 = \frac{1}{3} (1)^3 - \frac{1}{3} (0)^3 \\
= \frac{1}{3} - 0 = \frac{1}{3}.
\]

- Calculate \( \int_0^{\pi} \sin x \, dx \).

\( F(x) = -\cos x \) is an antiderivative of \( \sin x \), so

\[
\int_0^{\pi} \sin x \, dx = -\cos x \bigg|_0^{\pi} = (-\cos \pi) - (-\cos 0) \\
= (-(-1)) - (-1) \\
= 1 + 1 = 2.
\]