Lecture 20

Last time: integrals and calculating them, using FTC.

FTC:

I. $\int_a^x f(t) \, dt$ is an antiderivative of $f(x)$.
   
   i.e. $\frac{d}{dx} \int_a^x f(t) \, dt = f(x)$.

II. $\int_a^b f(x) \, dx = F(b) - F(a) = F(b) - F(a)$, where $F(x)$ is any antiderivative of $f(x)$.

Ex. $\frac{d}{dx} \int_3^x \cos (t^2) \, dt = \cos (x^2)$, \hfill \text{(FTC I)}$

Ex. $\int_0^{\pi} \sin x \, dx = 2$ \hfill \text{(last time)}

Ex. $\int_0^{\pi} \cos x \, dx = ?$ and what does it mean in terms of areas?

\[
\int_0^{\pi} \cos x \, dx = \sin x \bigg|_0^\pi = \sin(\pi) - \sin(0) = 0 - 0 = 0.
\]

\[
\text{i.e. the area of}
\]
\[
\text{blue, orange regions are equal, hence } A = A \quad (= 0)
\]

\[
\text{total integral is } \quad A - A = 0
\]

\[
\text{Rk could also look at } \int_0^{\pi} |\cos x| \, dx \text{ to get the actual area}
\]

\[
\text{To do this, break it up piecewise: } |\cos x| = \begin{cases} 
\cos x & 0 \leq x \leq \frac{\pi}{2} \\
-\cos x & \frac{\pi}{2} \leq x \leq \pi
\end{cases}
\]

\[
\text{so } \int_0^{\pi} |\cos x| \, dx = \int_0^{\pi/2} \cos x \, dx + \int_{\pi/2}^{\pi} -\cos x \, dx = ... = 1+1 = 2.
\]
Ex. Calculate \( \int_{\pi/4}^{\pi/3} \sec \theta \tan \theta \, d\theta \).

Use FTC II. \( \sec \Theta \) is an antiderivative of \( \sec \Theta + \tan \Theta \).

So: \( \int_{\pi/4}^{\pi/3} \sec \theta \tan \theta \, d\theta = \sec \Theta \bigg|_{\pi/4}^{\pi/3} = \sec \frac{\pi}{3} - \sec \frac{\pi}{4} \)

\[ = 2 - \sqrt{2} \]

Ex. Calculate \( \int_{-1}^{2} x^3 \, dx \) and interpret it as a difference of areas.

\[ \int_{-1}^{2} x^3 \, dx = \frac{1}{4} x^4 \bigg|_{-1}^{2} = \frac{1}{4} (2)^4 - \frac{1}{4} (-1)^4 \]

\[ = 4 - \frac{1}{4} = \frac{15}{4} \]

i.e. \( A_1 - A_2 = \frac{15}{4} \)

(Remark: if we use another antiderivative, we still get the same answer, e.g.

\[ \int_{-1}^{2} x^3 \, dx = \frac{1}{4} x^4 + 1 \bigg|_{-1}^{2} = \ldots = 5 - \frac{5}{4} = \frac{15}{4} \]

Ex. Is \( \int_{-1}^{1/2} \tan x \, dx \) positive, negative or zero?

\[ \int_{-1}^{1/2} \tan x \, dx = A_1 - A_2 < 0 \]

since \( A_2 \) is bigger than \( A_1 \).

Ex. Is \( \int_{-\pi/6}^{\pi/6} \tan x \, dx \) positive, negative or zero? It's zero.
General rule: integrals of symmetric functions

a) If \( f \) is odd, \( f(-x) = -f(x) \)
then \( \int_{-a}^{a} f(x) \, dx = 0 \)

b) If \( f \) is even, \( f(x) = f(-x) \)
then \( \int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx \)

\[
\text{Ex} \int_{-0.154}^{0.154} \frac{(\tan x) \left( x^6 + 29x^4 + \frac{105}{3} x^2 + 981.23 \right)}{x^{12} + 7 \cos (32x)} \, dx = 0
\]

\[
\text{Ex} \int_{\pi/6}^{\pi/2} \left( -\frac{3}{\sin^2 \Theta} + \Theta \right) \, d\Theta
\]

\[
= \int_{\pi/6}^{\pi/2} \left( -3 \csc^2 \Theta + \Theta \right) \, d\Theta
\]

\[
= 3 \cot \Theta + \frac{\Theta^2}{2} \bigg|_{\pi/6}^{\pi/3}
\]

\[
= \ldots = -2\sqrt{3} + \frac{\pi^2}{24}
\]

Indefinite integrals

Notation: \( \int f(x) \, dx \) means any antiderivative of \( f(x) \).
\[ \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \]
\[ \int \sin x \, dx = -\cos x + C \]
\[ \int \sec^2 x \, dx = \tan x + C \]
\[ \int \sec x \tan x \, dx = \sec x + C \]
\[ \int \csc x \cot x \, dx = -\csc x + C \]

\( E_x \)
Find \( \int (10x^4 + 6 \sec^2 x) \, dx. \)
\[ = 10 \left( \frac{x^5}{5} \right) + 6 \cdot \tan x + C = \frac{2x^5}{5} + 6 \tan x + C \]

\( E_\pi \)
Find \( \int_0^{\pi/4} (10x^4 + 6 \sec^2 x) \, dx \)
\[ = \left. 2x^5 + 6 \tan x \right|_0^{\pi/4} \]
\[ = \left[ 2 \left( \frac{\pi}{4} \right)^5 + 6 \tan \left( \frac{\pi}{4} \right) \right] - \left[ 2 \cdot 0^5 + 6 \cdot \tan (0) \right] \]
\[ = \frac{\pi^5}{512} + 6 \]
\[ = \frac{\pi^5}{512} + 6 \]

\( E_x \)
Find \( \int u^{\frac{2}{3}} \, du. \)
\[ n = \frac{2}{3} \quad n+1 = \frac{5}{3} \]
\[ \frac{u^{\frac{5}{3}}}{\frac{5}{3}} + C = \frac{3}{5} u^{\frac{5}{3}} + C \]

\( E_x \)
Find \( \int_1^8 u^{\frac{4}{3}} \, du \)
\[ = \frac{3}{5} \left[ u^{\frac{5}{3}} \right]_1^8 = \frac{3}{5} \left( 8^{\frac{5}{3}} \right) - \frac{3}{5} \left( 1^{\frac{5}{3}} \right) = \frac{3}{5} \left( 2^5 \right) - \frac{3}{5} (1) = \frac{3}{5} (32 - 1) \]
\[ = \frac{93}{5} \]
Net change

Given a function \( F(t) \), \( t = \text{time} \)
\( F'(t) \) is the rate of change of \( F(t) \).

\[
\int_a^b F'(t) \, dt = F(b) - F(a) = \text{net change of } F \text{ over time interval } [a, b].
\]

Ex
Water flows into a reservoir at the rate \((10t+6) \text{ ft}^3/\text{s}\). (\(t\) in sec)
The reservoir contains 400 \text{ ft}^3 of water at time \( t = 0 \).
How much does it contain at time \( t = 10 \text{ s} \)?

The net change from \( t = 0 \) to \( t = 10 \) is

\[
\int_0^{10} (10t+6) \, dt = 5t^2 + 6t \bigg|_0^{10}
\]

\[
= (5(10^2) + 6(10)) - (5(0^2) + 6(0))
\]

\[
= (500 + 60) - (0 + 0)
\]

\[
= 560 \text{ ft}^3
\]

So, the amount at time \( t = 10 \) is \( 400 + 560 = 960 \text{ ft}^3 \).

Ex
A capacitor is connected to a load that can charge or discharge it.
The current flowing into the cap. is \( Q'(t) = \sin(\pi t) + \frac{1}{2} \).
(\(Q(t)\) = charge of battery at time \( t \))

If the cap. starts with 10 units of charge at \( t = 0 \) (\( Q(0) = 10 \))
how much does it have at \( t = 6 \)?
\[ Q(b) - Q(0) = \int_0^b Q'(t) \, dt \]

\[ = \int_0^b \left( \sin(\pi t) + \frac{1}{2} \right) \, dt \]

\[ = -\frac{1}{\pi} \cos(\pi t) + \frac{t}{2} \bigg|_0^b \]

\[ = \left( -\frac{1}{\pi} \cos(6\pi) + 3 \right) - \left( -\frac{1}{\pi} \cos(0\pi) + 0 \right) \]

\[ = \left( -\frac{1}{\pi} + 3 \right) - \left( -\frac{1}{\pi} + 0 \right) \]

\[ = 3 \]

so \[ Q(b) = 3 + Q(0) = 3 + 10 = \frac{13}{2} \].

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A standard example of net charge: total displacement.

If \( s(t) = \) position along a line
\[ s'(t) = v(t) \] velocity

\[ v(t) > 0 : \text{moving to the right} \]
\[ v(t) < 0 : \text{"..." left} \]

Total displacement \[ s(b) - s(a) = \int_a^b v(t) \, dt = A_1 - A_2 + A_3 \]

Total distance (odometer reading) \[ \int_a^b |v(t)| \, dt = A_1 + A_2 + A_3 \]
A particle moves along a line with \( v(t) = t^2 - t - 6 \) m/s, from time \( t=1 \) to \( t=4 \).

a) What is the total displacement of the particle?

\[
\Delta s = s(4) - s(1) = \int_1^4 v(t) \, dt = \int_1^4 (t^2 - t - 6) \, dt
\]

\[
= \left. \frac{t^3}{3} - \frac{t^2}{2} - 6t \right|_1^4
\]

\[
= \text{...} = -\frac{9}{2} \quad \text{(i.e. } \frac{9}{2} \text{ m to the left, negative direction)}
\]

b) What is the total distance it travels?

\[
\int_1^4 |v(t)| \, dt
\]

\( v(t) = (t-3)(t+2) \)

\[
\begin{array}{ccc}
\text{ } & v>0 & v<0 & v>0 \\
\text{ } & \text{-2} & \text{3} & \text{ } \\
\end{array}
\]

\[
\int_1^4 |v(t)| \, dt = \int_1^3 |v(t)| \, dt + \int_3^4 |v(t)| \, dt
\]

\[
= \int_1^3 -v(t) \, dt + \int_3^4 v(t) \, dt = \text{...}
\]