Last time: Areas between curves

\[ y = g(x) \]
\[ y = f(x) \]

rectangle has area = \( \Delta x (f(x) - g(x)) \)

Sum all the rectangles, take limit \( \Delta x \to 0 \)

get total area: \( A = \int_a^b \, dx \, (f(x) - g(x)) \)

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Volumes (Ch 6.2)

Suppose we have some 3-d solid and we want to calculate its volume.

We chop the solid into slices:

volume of the slice = \( A(x) \cdot \Delta x \)

Add them all up, take \( \Delta x \to 0 \) limit.

Total volume of the solid: \( V = \int_a^b \, A(x) \, dx \)
Example: Calculate the volume of a ball of radius \(r\).

The ball is:
\[ x^2 + y^2 + z^2 \leq r^2 \]

Fixing a plane at constant \(x\):
\[ y^2 + z^2 \leq r^2 - x^2 \]

i.e. \((y, z)\) lie inside circle of radius \(\sqrt{r^2 - x^2}\).

So, the cross sections are circles, with area
\[ A(x) = \pi \left( \sqrt{r^2 - x^2} \right)^2 \]
i.e.
\[ A(x) = \pi (r^2 - x^2) \]

Volume of sphere:
\[ V = \int_{-r}^{r} A(x) \, dx \]
\[ = \int_{-r}^{r} \pi (r^2 - x^2) \, dx \]
\[ = \pi \left( r^2 x - \frac{1}{3} x^3 \right) \bigg|_{-r}^{r} \]
\[ = \pi \left( r^3 - \frac{1}{3} r^3 - (-r^3 + \frac{1}{3} r^3) \right) \]
\[ = \pi \left( \frac{2}{3} r^3 - (-\frac{2}{3} r^3) \right) \]
\[ = \pi \cdot \frac{4}{3} r^3 \]

A common type of solid: "solid of revolution" — take some region of the plane and revolve it around, say, the \(x\)-axis.

Example:

\[ y = x = f(x) \]
The cross-section of either of these solids at fixed $x$, is a circle of radius $f(x)$.
So the cross-section area is $A(x) = \pi \cdot f(x)^2$.

**Ex.** Find the volume of a solid obtained by revolving the region between $y = \sqrt{x}$, $y = 0$, $x = 2$ around the $x$-axis.

\[
V = \int_0^2 A(x) \, dx = \int_0^2 \pi \cdot (\sqrt{x})^2 \, dx = \int_0^2 \pi \cdot x \, dx = \pi \left. \frac{x^2}{2} \right|_0^2 = \pi (2^2) = 4\pi
\]

Can also revolve around e.g. the $y$-axis.

**Ex.** Find the volume of a solid obtained by revolving the region between

\[
x = y - y^2
\]

around the $y$-axis.

Intersects: \(x = 0\) \(\Rightarrow\) \(y - y^2 = 0\), i.e. \(y(1-y) = 0\), i.e. \(y = 0\) or \(y = 1\)

\((0, 0)\) \((0, 1)\)

Slice horizontally: i.e. look at cross-sections at fixed value of $y$.
The cross-section are circles, with radius depending on $y$ (\(f(y)\))
radius $= y - y^2$
area $= A(y) = \pi (y - y^2)^2$
We may also get cross-sections which are "washers"

**Ex.** Let $R$ be the region between $y = \sqrt{x}$ and $x = 2y$.
Find the volume of the solid obtained by rotating $R$ around the y-axis.

Intersect: $y = \sqrt{x} \rightarrow y^2 = x \rightarrow 2y = y^2$, i.e., $y^2 - 2y = 0$, i.e., $y(y-2) = 0$
so $y = 0$ or $y = 2$

$(0,0)$ $(4,2)$

Rotate about y-axis.
Cross section at fixed $y^1$

$x = 2y$  $x = y^2$

radius = $y^2$

radius = $2y$

$A(y) = \pi(2y)^2 - \pi(y^2)^2$

add up all the slices: $V = \int_0^2 A(y) \, dy$

$= \int_0^2 4y^2 - y^4 \, dy = \cdots = \frac{64}{15} \pi$