HW14 (last one) due next Friday
Midterm 3 next Tue — covers up to Lecture 22/HW13
My office hr: today as usual 4-5:30
extra next Monday 2:30-3:30

Last time: computing volumes of solids by integration

\[ \text{total volume} = \int_a^b A(x) \, dx \]

In examples from last time, cross sections were discs or washers — all made from circles.

One example that's different:

**Ex.** Calculate the volume of a solid whose base is the region between \( y=x, y=-x \) and \( x=2 \) and whose cross sections at fixed \( x \) are equilateral triangles.

\[ V = \int_0^2 A(x) \, dx \]

\[ A(x) = \text{area of equilateral triangle with side length } 2x \]

\[ \implies A(x) = \sqrt{3} x^2 \]
\[ V = \int_{0}^{2} \sqrt{3} x^2 \, dx = \frac{8\sqrt{3}}{3} \]

**Average values**

\[ P(t) = \text{price of coffee at time } t \]

How do we define the average price of coffee over the last year?

We know how to take average of a finite collection of numbers:

- average of \( \{2, 4\} = \frac{2+4}{2} = \frac{6}{2} = 3 \)
- \( \{1, 1, 1, 1, 7\} = \frac{1+1+1+1+7}{5} = \frac{11}{5} \)
- \( \{-1, 0, 2\} = \frac{-1+0+2}{3} = \frac{1}{3} \)
- \( \{y_1, y_2, \ldots, y_n\} = \frac{y_1 + y_2 + \ldots + y_n}{n} = \frac{1}{n} \sum_{i=1}^{n} y_i \)

So, to define the average value of a function \( f(x) \) on domain \([a, b] \):

\[ \int_{a}^{b} f(x) \, dx = \frac{1}{b-a} \cdot \left( \int_{a}^{b} f(x) \, dx \right) \]

\[ (\text{approximate average of } f \text{ over } [a, b]) = \frac{1}{b-a} \cdot \left( \int_{a}^{b} f(x) \, dx \right) \]
So, we define:

The average value of \( f(x) \) on the interval \([a, b]\) is

\[
\frac{1}{b-a} \int_a^b f(x) \, dx.
\]

Ex. Compute the average value of \( f(x) = \sin x \) on \([0, \pi]\).

\[
A = \frac{1}{\pi-0} \int_0^\pi \sin x \, dx
\]
\[
= \frac{1}{\pi} \left( -\cos x \big|_0^\pi \right)
\]
\[
= \frac{1}{\pi} \left( -(-1) - (-1) \right)
\]
\[
= \frac{2}{\pi}
\]

Ex. The average value of \( f(x) = c \) (constant)
on \([a, b]\) is:

\[
\frac{1}{b-a} \int_a^b c \, dx = \frac{1}{b-a} \left( c x \big|_a^b \right)
\]
\[
= \frac{1}{b-a} \cdot c (b-a)
\]
\[
= \frac{c}{a} \quad \text{(as expected)}
\]

Ex. The average value of \( \sin(x) \) over \([0, 2\pi]\) is

\[
\frac{1}{2\pi-0} \int_0^{2\pi} \sin x \, dx
\]
\[
= \frac{1}{2\pi} \left( -\cos x \big|_0^{2\pi} \right)
\]
\[
= \frac{1}{2\pi} \left( -1 - (-1) \right) = 0
\]
Ex The average value of \( f(x) = x \) over \([0, 10]\) is

\[
\frac{1}{10-0} \int_0^{10} x \, dx
\]

\[
= \frac{1}{10} \left( \frac{x^2}{2} \right)_0^{10}
\]

\[
= \frac{1}{10} \left( 100 - 0 \right)
\]

\[
= \frac{50}{10} = \frac{5}{2}
\]

as you would guess.

Ex The average value of \( f(x) = \sin^2 x \) over \([0, 2\pi]\) is

\[
\frac{1}{2\pi} \int_0^{2\pi} \sin^2 x \, dx = ?
\]

Use trig identity: \( \cos 2x = 2\sin^2 x - 1 \)

\[
\sin^2 x = \frac{1}{2} (\cos 2x + 1)
\]

so have

\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} (\cos 2x + 1) \, dx
\]

\[
= \cdots = \frac{1}{2\pi} \left( \pi \right) = \frac{1}{2}
\]
**Work**

A definition from physics:

if an object moves a distance $\Delta x$, 
acted on by a constant force $F$ ($F > 0$ for forces pushing in the direction $F < 0$ "..." "..." "..." "..." "..."")

we say the force does work on the object, 

$$ W = F \cdot \Delta x $$

**Ex** to lift a rock weighing 1 kg 
for a height $\Delta x = \frac{1}{2} m$ 
with constant speed, 

need a force $F = (1 \text{ kg}) \cdot (9.8 \text{ m/s}^2) = 9.8 \text{ N}$ 

the work is $W = F \cdot \Delta x = (9.8 \text{ N}) \cdot (\frac{1}{2} \text{ m}) = 4.9 \text{ J}$

What if the force $F$ is not constant?

Then we put 

$$ W = \int F \, dx $$

**Ex** A block is attached to a spring

When the block is at position $x$, 
the spring exerts a force 

$$ F = -kx $$

$k$ "spring constant"

When the block moves from $x = 0$ to $x = c$, how much work does the spring do on the block?

$$ W = \int_0^c F \, dx = \int_0^c (-kx) \, dx = -\frac{1}{2} kx^2 \bigg|_0^c = -\frac{1}{2} kc^2 $$
Why do we want to calculate the work?

Recall: if $F$ is the force on an object then $F = ma$.

Then the total work over some process is

$$W = \int_{x_0}^{x_1} F \, dx$$

$$= \int_{x_0}^{x_1} ma \, dx$$

$$= \int_{x_0}^{x_1} m \frac{dv}{dt} \, dx$$

$$= \int_{v_0}^{v_1} m \frac{dx}{dt} \, dv$$

$$= \int_{v_0}^{v_1} mv \, dv = \frac{1}{2}mv^2 \bigg|_{v_0}^{v_1} = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2$$