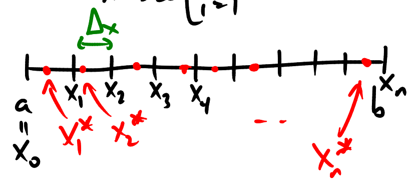
Lecture 30 | 12 Nov 2018

Last time: definite integral

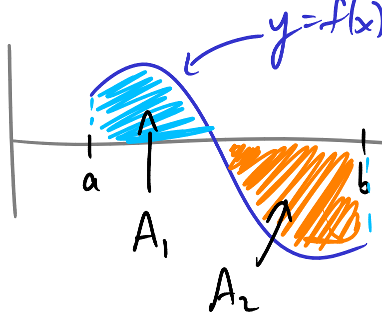
- its definition via Riemann sums

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i^*)\,\Delta x\right]$$



$x_i = x_0 + i\Delta x$

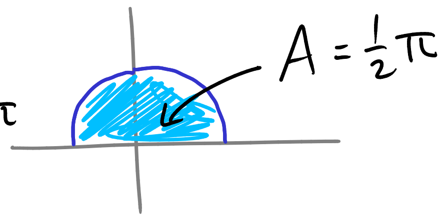
- its properties



$$\int_a^b f(x)\,dx = A_1 - A_2$$

- a few simple examples,

e.g. $\int_{-1}^{1}\sqrt{1-x^2}\,dx = \frac{1}{2}\pi$



How to actually calculate definite integrals?

Fundamental Theorem of Calculus

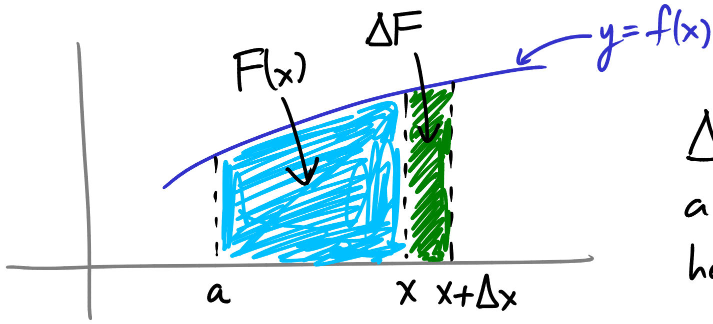
FTC I:

If $F(x) = \int_a^x f(t)\,dt$ then $F'(x) = f(x)$.

const. (a)

(ie $\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$.)

[Why?



ΔF is approx the area of a rectangle with width Δx height $f(x)$

ie $\Delta F \approx \Delta x \cdot f(x)$

ie $\frac{\Delta F}{\Delta x} \approx f(x)$

and as take $\Delta x \to 0$ that becomes $\frac{dF}{dx} = f(x)$]

Ex What is the derivative of $F(x) = \int_{-4}^{x} \sin t \, dt$?

$F'(x) = \sin x$ by FTC I.

Ex What is the deriv. of $F(x) = \int_4^{x^2} \cos t \, dt$?

Use chain rule: $\frac{d}{dx} \int_4^{x^2} \cos t \, dt$ $\qquad u = x^2$

$= \frac{d}{dx} \int_4^{u} \cos t \, dt$

$= \underbrace{\frac{du}{dx}} \cdot \underbrace{\frac{d}{du} \int_4^{u} \cos t \, dt}$

$= 2x \cdot \cos u$

$= \underline{2x \cdot \cos(x^2)}$

NB: the deriv of $\int_4^x \sin t \, dt$ didn't depend on the lower limit 4!

eg $\frac{d}{dx} \int_4^x \sin t \, dt = \sin x = \frac{d}{dx} \int_6^x \sin t \, dt$

So $\int_4^x \sin t \, dt$ and $\int_6^x \sin t \, dt$ must differ by a constant

It's indeed true: $\int_4^x \sin t \, dt = \int_6^x \sin t \, dt + \int_4^6 \sin t \, dt$ $\leftarrow C$

Ex $\frac{d}{dx}\left(\int_x^5 \sqrt{\sin t} \, dt \right) = ?$

by FTC I

$\frac{d}{dx}\left(\int_x^5 \sqrt{\sin t} \, dt \right) = \frac{d}{dx}\left(-\int_5^x \sqrt{\sin t} \, dt \right) = -\sqrt{\sin x}$

FTC II:

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{where } F(x) \text{ is } \underline{\text{any}} \text{ antiderivative of } f(x).$$

(exercise: deduce FTC II from FTC I!)

notation: $F(b)-F(a)$ is sometimes written $F\Big|_a^b$.

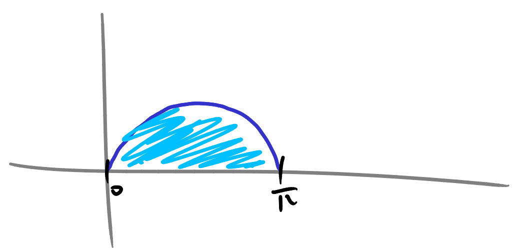
Ex · Calculate $\int_0^1 x^2\,dx$.

Use FTC II: $F(x) = \frac{1}{3}x^3$ is an antiderivative of x^2, so

$$\int_0^1 x^2\,dx = F(1) - F(0) = \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 = \underline{\underline{\frac{1}{3}}}.$$

or: $$\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 = \underline{\underline{\frac{1}{3}}}.$$

Ex · Calculate $\int_0^\pi \sin x\,dx$.



$F(x) = -\cos x$

$$\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = (-\cos\pi) - (-\cos 0) = -(-1) - (-1) = 2.$$

or, if we use $F(x) = -\cos x + 16$

$$-\cos x + 16\Big|_0^\pi = (-\cos\pi + 16) - (-\cos 0 + 16) = (1+16) - (-1+16) = 17 - 15 = \underline{\underline{2}}$$

Ex $\displaystyle\int_{\pi/4}^{\pi/3} \sec\theta \tan\theta \, d\theta = ?$

$\sec\theta$ is an antideriv. of $\sec\theta \tan\theta$

$\left(\text{b/c } \frac{d}{d\theta} \sec\theta = \sec\theta \tan\theta \right)$

so, $\displaystyle\int_{\pi/4}^{\pi/3} \sec\theta \tan\theta \, d\theta = \sec\theta \Big|_{\pi/4}^{\pi/3} = \sec\pi/3 - \sec\pi/4$

$= \underline{\underline{2 - \sqrt{2}}}$