Housekeeping: my wife’s baby due 23 Oct

Last time: double integrals over general regions

Sequences (Ch 11.1)

A sequence is an ordered list of numbers

\[ \{a_n\} = a_1, a_2, a_3, a_4, \ldots, a_{100}, \ldots \]

Ex

\[ a_n = n : \ 1, 2, 3, 4, 5, \ldots, 100, \ldots \]

\[ a_n = (-1)^n : \ -1, 1, -1, 1, -1, \ldots, 1, \ldots \]

\[ a_n = \frac{1}{n^2 + 1} : \ \frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \ldots, \frac{1}{10001}, \ldots \]

\( a_n = \) the closing price of DJIA on the \( n \)th trading day of this year:

10784.6, 10812.9, 10800.3, \ldots
\( a_n = \) the \( n \)th term of the Fibonacci sequence (i.e. \( a_1 = 1, a_2 = 1, a_{n+2} = a_n + a_{n+1} \))

1, 1, 2, 3, 5, 8, 13, 21, ...

\[ a_1, a_2, a_3 = a_1 + a_2, a_4 = a_2 + a_3 \]

\[ a_n = \frac{1}{n} \cos \left( \frac{n \pi}{2} \right) \]

0, -\( \frac{1}{2} \), 0, \( \frac{1}{4} \), 0, -\( \frac{1}{6} \), 0, \( \frac{1}{8} \), 0, -\( \frac{1}{10} \), 0, \( \frac{1}{12} \), 0, -\( \frac{1}{14} \), ...

Ex Consider the sequence 3, 8, 13, 18, 23, 28, ...
where each term differs from the previous one by 5.
What is \( a_n \)?

\[ a_n = 5n - 2 \]

Ex \[ a_n = n! \quad [n! = (n)(n-1)(n-2)\ldots(1) \quad \text{e.g.} \quad 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24] \]

1, 2, 6, 24, 120, 720, ...

Fundamental question about a sequence \( \{a_n\} \): does it converge?

\[ \lim_{n \to \infty} a_n = L \]

We say \( \{a_n\} \) converges to \( L \) if by making \( n \) big enough we can make \( a_n \) as close to \( L \) as we like. (For more formal def, see p. 692 of text, "Def 2")

We say \( \{a_n\} \) diverges if it does not converge to any \( L \).
\[ a_n = n : \]
\[ \text{diverges} \]

\[ a_n = (-1)^n : \]
\[ \text{diverges} \]

\[ a_n = \frac{1}{n} : \]
\[ \text{converges to 0} \]

We also express this as \( \lim_{n \to \infty} a_n = 0 \), i.e. \( \lim_{n \to \infty} \frac{1}{n} = 0 \).

This might remind you of the fact that \( \lim_{x \to \infty} \frac{1}{x} = 0 \).

Indeed: if our sequence is given by a function, \( a_n = f(n) \), and
\[ \lim_{x \to \infty} f(x) = L, \text{ then } \lim_{n \to \infty} a_n = L, \text{ i.e. } \{a_n\} \text{ converges to } L. \]
Ex Does \( a_n = \frac{\ln n}{n} \) converge?

Look at \( f(x) = \frac{\ln x}{x} \). \( \lim_{x \to \infty} \frac{\ln x}{x} = \frac{\infty}{\infty} \), use L'Hôpital:

\[
= \lim_{x \to \infty} \frac{(1/x)}{1} = \lim_{x \to \infty} \frac{1}{x} = \frac{1}{\infty} = 0
\]

So, \( a_n = \frac{\ln n}{n} \) converges to 0.

Ex Does \( a_n = \frac{\ln n}{n} - 4 \) converge, and to what?

\[
\lim_{n \to \infty} \left( \frac{\ln n}{n} - 4 \right) = \lim_{n \to \infty} \left( \frac{\ln n}{n} \right) - \lim_{n \to \infty} (4)
\]

\[
= 0 - 4 = -4
\]

\( \Rightarrow \) Yes, it converges to -4

This was an example of "Limit Laws for Sequences":

If \( a_n, b_n \) are 2 sequences which converge, then

\[
\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n
\]

\[
\lim_{n \to \infty} c \cdot a_n = c \cdot \lim_{n \to \infty} a_n
\]

\[
\lim_{n \to \infty} a_n \cdot b_n = \left( \lim_{n \to \infty} a_n \right) \cdot \left( \lim_{n \to \infty} b_n \right)
\]

\[
\lim_{n \to \infty} a_n^k = \left( \lim_{n \to \infty} a_n \right)^k
\]
Ex. Does $a_n = \sin \left( \frac{\pi n}{1 + 4n} \right)$ converge?

Look at $f(x) = \sin \left( \frac{\pi x}{1 + 4x} \right)$

$$\lim_{x \to \infty} \sin \left( \frac{\pi x}{1 + 4x} \right) = \lim_{x \to \infty} \sin \left( \frac{\pi x \cdot \frac{1}{x}}{(1 + 4x) \cdot \frac{1}{x}} \right) = \lim_{x \to \infty} \sin \left( \frac{\pi}{\frac{1}{x} + 4} \right)$$

$$= \sin \left( \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2}$$

So $a_n$ conv. to $\frac{\sqrt{2}}{2}$.