Last time: definition of $\int_a^b f(x) \, dx$

Now we learn a much easier way to calculate integrals. (See S.3)

Fundamental Theorem of Calculus I:

If $F(x) = \int_a^x f(t) \, dt$
then $F'(x) = f(x)$. \[\text{[ie } \int_a^x f(t) \, dt \text{ is an antiderivative of } f(x).]\]

Examples.

• What is the derivative of $F(x) = \int_{-4}^x \sin t \, dt$?

By FTC I, $F'(x) = \sin x$.

• What is the derivative of $F(x) = \int_4^{x^2} \cos t \, dt$?

[Careful — not just $\cos(x^2)$!]

Apply chain rule: $\frac{d}{dx} \int_4^{x^2} \cos t \, dt$

$= \frac{d}{dx} \int_4^{u} \cos t \, dt$

$= \frac{du}{dx} \cdot \frac{d}{du} \int_4^{u} \cos t \, dt$

$= 2x \cdot \cos(u)$

$= 2x \cdot \cos(x^2)$
Suppose \( \int_{-1}^{x} f(t) \, dt = \frac{1}{x^2 + 1} \). What is \( f(2) \)?

Use FTC I: apply \( \frac{d}{dx} \) to both sides.

\[
\frac{d}{dx} \int_{-1}^{x} f(t) \, dt = \frac{d}{dx} \frac{1}{x^2 + 1}.
\]

\[
f(x) = -\frac{2x}{(x^2 + 1)^2}
\]

\[
f(2) = -\frac{4}{25}
\]

\[\text{Ex: } \frac{d}{dx} \int_{x}^{5} f(x) \, dx = \frac{d}{dx} \left( -\int_{x}^{5} f(x) \, dx \right) = -f(x)\]

**Fundamental Theorem of Calculus II:**

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a) \text{ where } F \text{ is any antiderivative of } f.
\]

**Notation:** \( F(b) - F(a) \) is also written as \( F|_{a}^{b} \) or \( [F]_{a}^{b} \).

**Examples:**

- Calculate \( \int_{0}^{1} x^2 \, dx \).
  
  Use FTC II: \( F(x) = \frac{1}{3} x^3 \) is an antiderivative of \( x^2 \), so
  
  \[
  \int_{0}^{1} x^2 \, dx = \frac{1}{3} x^3 \bigg|_{0}^{1} = \frac{1}{3} (1^3) - \frac{1}{3} (0^3) = \frac{1}{3}
  \]
• Calculate $\int_0^\pi \sin x \, dx$.

$F(x) = -\cos x + C$ is an antiderivative of $\sin x$, so

$$\int_0^\pi \sin x \, dx = -\cos x \bigg|_0^\pi = (-\cos \pi + C) - (-\cos 0 + C)$$

$$= -(-1) - (-1) + C - C = 1 + 1 = 2$$

area = 2

• Calculate $\int_{\pi/4}^{\pi/3} \sec \theta \tan \theta \, d\theta$.

$\sec \theta$ is an antiderivative of $\sec \theta \tan \theta$, so

$$\int_{\pi/4}^{\pi/3} \sec \theta \tan \theta \, d\theta = \sec \theta \bigg|_{\pi/4}^{\pi/3} = \sec \frac{\pi}{3} - \sec \frac{\pi}{4}$$

$$= 2 - \sqrt{2}$$

• Calculate $\int_{-1}^{2} x^3 \, dx$ and interpret it as a difference of areas.

$$\int_{-1}^{2} x^3 \, dx = \frac{x^4}{4} \bigg|_{-1}^{2} = \frac{2^4}{4} - \frac{(-1)^4}{4} = \frac{15}{4}$$

$A_1 - A_2 = \frac{15}{4}$
• Calculate \( \int_{\pi/6}^{\pi/3} (-\frac{3}{\sin^2 \theta} + \theta) \, d\theta \).
  \[
  = \int_{\pi/6}^{\pi/3} (-3 \cos^2 \theta + \theta) \, d\theta \\
  = 3 \cot \theta + \frac{\theta^2}{2} \bigg|_{\pi/6}^{\pi/3} \\
  = ... = -2\sqrt{3} + \frac{\pi^2}{24}
  
  \]

• Calculate \( \int_{-2}^{1} 3 + u^4 \, du \).
  \[
  = 3u + \frac{1}{5}u^5 \bigg|_{-2}^{1} \\
  = \left[ 3(-2) + \frac{1}{5}(-2)^5 \right] - \left[ 3(1) + \frac{1}{5}(1)^5 \right] \\
  = -6 - \frac{32}{5} - 3 - \frac{1}{5} = -\frac{78}{5}
  
  \]

An example that belongs in the previous lecture:

• If \( \int_{1}^{3} f(x) \, dx = 4 \)
  and \( \int_{3}^{7} f(x) \, dx = 16 \)

  What is \( \int_{1}^{7} 3f(x) \, dx ? \)
  \[
  = 3 \int_{1}^{7} f(x) \, dx \\
  = 3 \left( \int_{1}^{3} f(x) \, dx + \int_{3}^{7} f(x) \, dx \right) \\
  = 3(4 + 16) = 60
  
  \]