Lecture 24

Housekeeping: Exam 2 Tue Apr 6 7-9pm WEL 1.316

Last time: improper integrals

So far we always looked at functions \( f(x) \)

\[ \text{only one variable} \]

In reality things often depend on more than one variable.

e.g. the heat index depends on temperature and humidity

\[ f(x,y) \]

\[ x \quad y \]

or the elevation depends on longitude and latitude

the price of cheese depends on supply and demand

Partial derivatives (Ch 15.3)

If we have \( f(x,y) \) we can ask:

- How does \( f \) change if we vary \( x \) a little while holding \( y \) constant?

Define

\[
\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}
\]

[To calculate it, just treat \( y \) as constant and differentiate like usual with resp to \( x \).]

Ex If \( f(x,y) = x^2 \sin(y) \) then \[ \frac{\partial f}{\partial x} = 2x \sin(y) \]

Ex If \( f(x,y) = x^2 \sin(y) \) then \[ \frac{\partial f}{\partial x} \] at \((x,y) = (1, \frac{\pi}{2})\) is \( 2 \cdot 1 = 2 \)
Similarly, \[ \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h} \]

(hold x constant and differentiate w/rpect to y).

**Ex.** Say \( f(x, y) = 4x^2y + 7\sin(x) \)

\[ \frac{\partial f}{\partial x} = 8xy + 7\cos(x) \quad \frac{\partial f}{\partial y} = 4x^2 \]

Can also look at \( 2^{nd} \) deriv:

\[ \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} (8y - 7\sin(x)) = 8y \]

\[ \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} (8x) = 8x \]

\[ \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} (8x^3) = 24x^2 \]

\[ \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} (4x^2) = 8x \]

\[ \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \]

**Notice:** \( \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} ! \)

This always happens (whenever both are continuous).
Example: \( f(x,y) = \sin(xy) \)

\[
\begin{align*}
\frac{\partial f}{\partial x} &= y \cos(xy) \\
\frac{\partial f}{\partial y} &= x \cos(xy) \\
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} (y \cos(xy)) \\
&= y \cdot y(- \sin(xy)) \\
&= -y^2 \sin(xy) \\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} (x \cos(xy)) \\
&= x \cdot x(- \sin(xy)) \\
&= -x^2 \sin(xy) \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{2}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{2}{\partial x} (x \cos(xy)) = 1 \cdot \cos(xy) + x \cdot (-y \sin(xy)) \\
&= \cos(xy) - xy \sin(xy)
\end{align*}
\]

\[\text{Also,} \quad \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \cos(xy) - xy \sin(xy) \]

because \( \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \)

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Picturing the partial derivatives:

Say \( f = e^{-x^2 - y^2} \)

\[
\begin{align*}
\frac{\partial f}{\partial x} &= -2x e^{-x^2 - y^2} \\
\frac{\partial f}{\partial y} &= -2y e^{-x^2 - y^2}
\end{align*}
\]

So at \((x,y)=(1,0)\):

\[
\begin{align*}
\frac{\partial f}{\partial x} &= -\frac{2}{e} < 0 \\
\frac{\partial f}{\partial y} &= 0
\end{align*}
\]

i.e. \( f \) decreasing \( \implies \frac{\partial f}{\partial x} < 0 \)

\( \uparrow \): neither uphill nor downhill \( \implies \frac{\partial f}{\partial y} = 0 \)

See this from the "contour map" of \( f \):

(lighter color = larger value of \( f \))
3-d picture of $f$: \((\text{height} = f(x,y))\)

Notation: Write $f_x$ for $\frac{\partial f}{\partial x}$

$f_y$ for $\frac{\partial f}{\partial y}$

$f_{xx}$ for $\frac{\partial^2 f}{\partial x^2}$

$f_{xy}$ for $\frac{\partial^2 f}{\partial x \partial y}$

$f_{yx}$ for $\frac{\partial^2 f}{\partial y \partial x}$

$f_{yy}$ for $\frac{\partial^2 f}{\partial y^2}$

[so $f_{xy} = f_{yx}$]

[etc. for higher derivatives]