Ex (from Lecture 34):

For what $p$ does the series
\[ \sum_{n=7}^{\infty} \left( \frac{n+1}{n^{5p}} \right) \cos(2\pi n) \]
converge?

First observation: $\cos(2\pi n) = 1$. So the series is really
\[ \sum_{n=7}^{\infty} \frac{n+1}{n^{5p}} \]

Now at large $n$ this would go \( \sim \frac{n}{n^{5p}} = \frac{1}{n^{5p-1}} \)

So try the Limit Comparison Test: using
\[ a_n = \frac{n+1}{n^{5p}}, \quad b_n = \frac{1}{n^{5p-1}} \]

To see if the test applies:
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left( \frac{n+1}{n^{5p}} \right) \cdot \left( \frac{1}{n^{5p-1}} \right) = \lim_{n \to \infty} \left( \frac{n+1}{n^{5p}} \cdot \frac{1}{n^{5p-1}} \right) = \lim_{n \to \infty} \left( \frac{n+1}{n^{10p-1}} \right)
\]

So the test applies: $\sum a_n$ converges if and only if $\sum b_n$ converges.

$\sum b_n = \sum \frac{1}{n^{5p-1}}$: use $p$-test — converges if $5p-1 > 1$

i.e. $p > \frac{2}{5}$
So finally, $\sum a_n$ converges if $p > \frac{2}{5}$
 diverges if $p \leq \frac{2}{5}$

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**Absolute Convergence**

$\sum a_n$

Call $\sum a_n$ “absolutely convergent” if $\sum |a_n|$ is convergent.

**Ex**

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \ldots$

$[a_n = \frac{(-1)^n}{n^2}]$

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (by p-test, $p = 2 > 1$)

So $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

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**Fact:** If $\sum a_n$ is absolutely convergent
then $\sum |a_n|$ is convergent.

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If $\sum a_n$ is convergent but $\sum |a_n|$ is not absolutely convergent, then we call $\sum a_n$ conditionally convergent.

**Ex** $\sum (-1)^n \cdot \frac{1}{n}$ is convergent (by alt. series test)
But $\sum (-1)^n \cdot \frac{1}{n}$ is not absolutely convergent
(because $\sum |(-1)^n \frac{1}{n}| = \sum \frac{1}{n}$ is divergent (by p-test))

So $\sum (-1)^n \frac{1}{n}$ is conditionally convergent

So have 3 possibilities:

- absolutely convergent
- conditionally convergent
- divergent

Ex: $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$

Has both positive and negative terms:

$+, -, -, +, +, +, +, -, ...$

Not alternating.

Is it absolutely convergent? Look at $\sum_{n=1}^{\infty} \left| \frac{\cos(n)}{n^2} \right|

We know $\left| \frac{\cos(n)}{n^2} \right| = \frac{\cos(n)}{n^2} \leq \frac{1}{n^2}$

And we know $\sum \frac{1}{n^2}$ converges (p-test)

So $\sum \left| \frac{\cos(n)}{n^2} \right|$ converges by Comparison Test

So $\sum \frac{\cos(n)}{n^2}$ converges absolutely
So \( \sum \frac{\cos(n)}{n^2} \) converges.

\[
\text{Ex} \quad \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}
\]

This is alternating series with \( b_n = \frac{1}{\ln n} \).

So by alternating series test, it converges.

Does it converge absolutely?

i.e. \( \sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln n} \) converge?

\[
\frac{1}{\ln n} > \frac{1}{n} \quad \text{and} \quad \sum \frac{1}{n} \text{ diverges} \]

\( \text{so} \ \sum \frac{1}{\ln n} \text{ diverges by Comparison Test.} \)

So \( \sum (-1)^n \frac{1}{\ln n} \) converges conditionally.
Ratios Test

1) If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1 \)
   then \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent.

2) If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1 \) (or \( 0 \))
   then \( \sum_{n=1}^{\infty} a_n \) is divergent.

(If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \) then the test is inconclusive.)

\[
\text{Ex} \sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}. \ \text{Ratio test}: \ a_n = (-1)^n \frac{n^3}{3^n}
\]
\[
a_{n+1} = (-1)^{n+1} \frac{(n+1)^3}{3^{n+1}}
\]
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(n+1)^3}{3^{n+1}} \frac{3^n}{n^3} = \lim_{n \to \infty} \frac{3^n}{n^3} \frac{(n+1)^3}{3^{n+1}} = \lim_{n \to \infty} \frac{1}{3} \left(1 + \frac{1}{n}\right)^3 = \frac{1}{3}
\]

Since \( L = \frac{1}{3} < 1 \), \( \sum a_n \) converges absolutely by Ratio Test.