Last time: Absolute and conditional convergence

Ratio Test

Ex: \( \sum_{n=1}^{\infty} \frac{n^n}{n!} \) \( a_n = \frac{n^n}{n!} \)

Ratio Test: look at \( \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} \)

\[
\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} = \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)!}
\]

\[
= \frac{(n+1) \cdot (n+1)^n}{n^n} \cdot \frac{n!}{(n+1)n!}
\]

\[
= \frac{(n+1)^n}{n^n} = \left( \frac{n+1}{n} \right)^n = (1 + \frac{1}{n})^n
\]

So \( \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} (1 + \frac{1}{n})^n = e \).

Since \( e > 1 \), Ratio Test says \( \sum \frac{n^n}{n!} \) diverges.

Ex: \( \sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2} \) \( a_n = \frac{\sqrt{n}}{1+n^2} \)

Suppose we try Ratio Test on this:
\[
\frac{|a_{n+1}|}{|a_n|} = \frac{\sqrt{n+1}/(n+1)^2}{\sqrt{n}/(n+1)^2} = \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{1+n^2}{1+(n+1)^2}
\]

and \(\lim_{n \to \infty} \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{1+n^2}{1+(n+1)^2} = 1\) (skipped simplification steps here)

So the Ratio Test is inconclusive here.

(Could see that this \(\sum a_n\) converges using Limit-Comp Test, with \(b_n = \frac{1}{n^{3/2}}\).

\[
\text{Root Test}
\]

\text{. If } \lim_{n \to \infty} \sqrt[n]{|a_n|} = L < 1

\text{Then } \sum_{n=1}^{\infty} a_n \text{ is absolutely convergent.}

\text{. If } \lim_{n \to \infty} \sqrt[n]{|a_n|} = L > 1 \text{ (or } = \infty) \text{)

\text{Then } \sum_{n=1}^{\infty} a_n \text{ is divergent.}

\left[ \text{If } \lim_{n \to \infty} \sqrt[n]{|a_n|} = 1 \text{ then the Root Test is inconclusive.} \right]
Ex. \[ \sum_{n=1}^{\infty} \left( \frac{2n+3}{3n+2} \right)^n \]

\[ a_n = \left( \frac{2n+3}{3n+2} \right)^n \]

Root Test: \[ \sqrt[n]{|a_n|} = \sqrt[n]{\left( \frac{2n+3}{3n+2} \right)^n} = \frac{2n+3}{3n+2} \]

and \[ \lim_{n \to \infty} \frac{2n+3}{3n+2} = \frac{2}{3} < 1 \]

So \[ \sum_{n=1}^{\infty} \left( \frac{2n+3}{3n+2} \right)^n \] converges (absolutely)

Strategy for Testing Series (Ch 12.7)

Classify the series according to its form.

1) \[ \sum \frac{1}{n^p} : \text{ p-test.} \]

2) \[ \sum ar^{-1} \text{ or } \sum ar^n : \text{ geometric series: converges if } |r|<1 \text{ diverges if } |r| \geq 1 \]

3) If the series looks similar to a p-series or geom series:
   - try comparison or limit-comparison (picking \( b_n \) to be the p-series or geometric series).
   - (If the series has some negative terms then apply this method instead to \( \sum |a_n| \) — i.e. test for absolute convergence.)

4) If you can see easily that \( \lim_{n \to \infty} a_n \neq 0 \), use Test For Divergence.
5) If the series is \( \sum (-1)^n b_n \) or \( \sum (-1)^{n+1} b_n \)
   try Alternating Series Test.

6) If the series involves factorials (or other products involving n terms, including \( k^n \)) — try Ratio Test.
   [But not for series where \( a_n \) is just rational function — Ratio Test will be inconclusive for those]

7) If \( a_n = \text{(something)}^n \) try Root Test.

8) If \( a_n = f(n) \) and you know how to do \( \int_1^\infty f(x) \, dx \)
   [and \( f(x) \) is decreasing for large enough \( x \)]
   try Integral Test.

\[ \sum (\frac{n^2+4}{3n^2+7n})^{3n} \]

\[ a_n = \left(\frac{n^2+4}{3n^2+7n}\right)^{3n} \quad \text{use Root Test} \quad \sqrt[n]{\left(\frac{n^2+4}{3n^2+7n}\right)^{3n}} \]
\[ = \left(\frac{n^2+4}{3n^2+7n}\right)^3 \]

\[ \ldots \]
\[ \sum \frac{n+8}{2n+1} : \text{ use Test For Divergence} \]

\[ \sum n^2e^{-n^3} : \text{ use Integral Test} \]
with \( f(x) = \frac{2^x}{e^{-x^3}} \)

\[ \sum (-1)^n \frac{n^3}{n^4+1} \]
[and if we want to see whether it's absolutely convergent, use Limit-Comparison with \( b_n = \frac{1}{n} \)]

\[ \sum \frac{2^k}{k!} : \text{ use Ratio Test} \]

\[ \sum n \sin \left( \frac{1}{n} \right) : \text{ use Test For Divergence} \]

\[ \sum \frac{1}{2+3^n} : \text{ use Comparison or Limit-Comparison} \]
with \( b_n = \frac{1}{3^n} \)

\[ \sum (-1)^j \frac{\sqrt{j}}{j+5} : \text{ use Alt. Series Test} \]