Housekeeping:
- UT Learning Center exam review tonight and tomorrow 6-8pm
- Course Instructor surveys now open at https://utdirect.utexas.edu/diia/ecis/
- Exam 3 next Tuesday 7-9pm

A quick comment about sequences vs series:
Take the sequence \( a_n = \frac{1}{n} \)
We may ask 2 different questions about this sequence.

Q1) Does the sequence \( \{a_n\} \) converge?
A1) Yes: \( \lim_{n \to \infty} \frac{1}{n} = 0 \).

Q2) Does the series \( \sum_{n=1}^{\infty} a_n \) converge?
A2) No: \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges by p-test.

Last time: Power series \( \sum_{n=0}^{\infty} c_n (x-a)^n \)

Functions As Power Series (Ch 12.9)
Remember the formula \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \) (geometric series
(for \( |x| < 1 \))
(first term = 1
common ratio = \( x \))
Ex: \[ \frac{1}{0.7} = \frac{1}{1-0.3} = \sum_{n=0}^{\infty} (0.3)^n \]
\[ = 1 + 0.3 + (0.3)^2 + (0.3)^3 + \ldots \]
\[ = 1 + 0.3 + 0.09 + \ldots \]
\[ \approx 1.4 \]

Ex: Find a representation of the function \( \frac{1}{1+x^2} \)
as a power series, and its interval of convergence, radius of convergence.

\[ \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n (x^2)^n \]
\[ = \sum_{n=0}^{\infty} (-1)^n x^{2n} \]

To find interval of convergence: could use Ratio Test. But here there’s shortcut: this is a geometric series with common ratio \( r = -x^2 \).

So it converges if and only if \( |r| < 1 \)

i.e. \( |-x^2| < 1 \)

i.e. \( |x|^2 < 1 \)

i.e. \( |x| < 1 \)

So the interval of convergence is \( (-1, 1) \). Radius of conv. \( R = 1 \).
Ex. Write $\frac{1}{x+7}$ as a power series.

$$\frac{1}{x+7} = \frac{1}{7} \left( \frac{1}{\frac{x}{7} + 1} \right)$$

$$= \frac{1}{7} \left( \frac{1}{1 - (-\frac{x}{7})} \right)$$

$$= \frac{1}{7} \left( \sum_{n=0}^{\infty} (-\frac{x}{7})^n \right)$$

$$= \frac{1}{7} \sum_{n=0}^{\infty} \frac{(-1)^n}{7^n} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{7^{n+1}} x^n \quad \text{[power series centered at 0]}

\[\text{We might also have written } \frac{1}{x+7} = \frac{1}{1 - (-6 - x)}\]

\[\text{and then get } \frac{1}{x+7} = \sum_{n=0}^{\infty} (-6 - x)^n = \sum_{n=0}^{\infty} (-1)^n (x+6)^n\]

\[\text{That's another power series for the same function, centered at } -6.\]

Ex. Write $\frac{x^4}{x+7}$ as a power series (centered at 0).

$$\frac{x^4}{x+7} = x^4 \cdot \frac{1}{x+7} = x^4 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{7^{n+1}} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{7^{n+1}} x^{n+4}$$

\[\text{from the previous example}\]
Could also rewrite this: let \( m = n + 4 \) (so \( n = m - 4 \))

then the above also equals

\[
\sum_{m=4}^{\infty} \frac{(-1)^{m-4}}{7^{m-3}} x^m
\]

\[
= \sum_{m=4}^{\infty} (-1)^m \frac{(-1)^{-q}}{7^{m-3}} x^m
\]

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Fact: If we have a power series for \( f(x) \),

\[ f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \]

Then also:

\[
\frac{d}{dx} f(x) = \frac{d}{dx} \sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=1}^{\infty} c_n \cdot n (x-a)^{n-1}
\]

\[
\int f(x) \, dx = \int \sum_{n=0}^{\infty} c_n (x-a)^n \, dx = \left( \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \right) + C
\]

Both of these new series have the same radius of convergence as the original one.

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Ex: Express \( \frac{1}{(1-x)^2} \) as a power series, find its radius of convergence (centered at \( x = 0 \))

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{(radius of conv = 1)}
\]

\[
\frac{d}{dx} \left( \frac{1}{1-x} \right) = \frac{d}{dx} \sum_{n=0}^{\infty} x^n
\]
\[
\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \quad \text{(also with radius of conv = 1)}
\]

Ex. Express \(\ln(1-x)\) as a power series (centered at 0).

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n
\]

\[
\int \frac{1}{1-x} \, dx = \int \sum_{n=0}^{\infty} x^n \, dx
\]

\[-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C\]

\[
\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} - C
\]

To determine \(C\): plug in \(x=0\), then the eq. becomes:

\[
\ln(1) = 0 - C
\]

\[0 = 0 - C \quad \text{so} \quad C = 0
\]

So we get

\[
\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}
\]

Also can rewrite this: set \(m = n+1\), then

\[
\ln(1-x) = -\sum_{m=1}^{\infty} \frac{x^m}{m}
\]