Last time: **partial derivatives**

\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2} &= f_{xx} \\
\frac{\partial^2 f}{\partial y^2} &= f_{yy} \\
\frac{\partial^2 f}{\partial y \partial x} &= f_{yx}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial f}{\partial x} &= f_x \\
\frac{\partial f}{\partial y} &= f_y
\end{align*}
\]

These two are equal!!

How about integrating a function of 2 variables?

![Graph of \( z = f(x, y) \)](image)

**Q:** What is the total volume under the graph (i.e., the volume between the graph of \(z = f(x, y)\) and the xy-plane) \(z = 0\)?

Cut by planes at fixed \(y\): \(V = \int_0^2 A(y) \, dy\)

\(A(y) = \int_0^3 f(x, y) \, dx\) gives the cross section area

\(\therefore V = \int_0^2 \left[ \int_0^3 f(x, y) \, dx \right] \, dy\)
Ex. Suppose \( f(x, y) = 4xy + 3x^2 \)

Then \( V = \int_0^2 \left[ \int_0^2 (4xy + 3x^2) \, dx \right] \, dy \)
\[= \int_0^2 \left( 2x^2 y + x^3 \right|_{x=0}^{x=3} \right) \, dy \]
\[= \int_0^2 (18y + 27) \, dy \]
\[= 9y^2 + 27y \big|_{y=0}^{y=2} \]
\[= 36 + 54 = 90 \]

We could also try doing the \( \int \)'s in the other order:

\( V = \int_0^3 \left[ \int_0^3 4xy + 3x^2 \, dy \right] \, dx \)
\[= \int_0^3 \left( 2xy^2 + 3x^2 y \big|_{y=0}^{y=2} \right) \, dx \]
\[= \int_0^3 8x + 6x^2 \, dx \]
\[= 4x^2 + 2x^3 \big|_0^3 = 36 + 54 = 90 \]

This gives the same answer — can choose either order \("Fubini's Theorem"")

\[\int_a^b \left[ \int_c^d f(x, y) \, dx \right] \, dy = \int_c^d \left[ \int_a^b f(x, y) \, dx \right] \, dy\]

We also write this as \( \iiint_{R} f(x, y) \, dA \)
If \( R = \{1 \leq x \leq 2, \ 0 \leq y \leq \pi\} \)

and \( f(x,y) = y \sin(xy) \)

what is \( \iint_R f(x,y) \, dA \)?

It is 
\[
\int_0^\pi \left[ \int_1^2 y \sin(xy) \, dx \right] \, dy
\]

(Or: 
\[
\int_1^2 \left[ \int_0^\pi y \sin(xy) \, dy \right] \, dx,
\]
but that's harder to calculate)

Now,
\[
\int_1^2 y \sin(xy) \, dx = y \cdot \int_1^2 \sin(xy) \, dx
\]
\[
= y \cdot \left[ -\frac{1}{y} \cos(xy) \right]_{x=1}^{x=2}
\]
\[
= -\cos(xy) \bigg|_{x=1}^{x=2}
\]
\[
= -\cos(2y) + \cos(y)
\]

So,
\[
\int_0^\pi \left[ \int_1^2 y \sin(xy) \, dx \right] \, dy
\]
\[
= \int_0^\pi (-\cos(2y) + \cos(y)) \, dy
\]
\[
= \left( -\frac{1}{2} \sin(2y) + \sin(y) \right) \bigg|_{y=0}^{y=\pi}
\]
\[
= 0
\]
Find the volume of the solid which lies under the graph of

\[ z = f(x, y) = 4 + x^2 - y^2 \]

and over the rectangle \[-1 \leq x \leq 1\]
\[ 0 \leq y \leq 2 \] (call this rectangle \( R \))

\[ V = \iint_R f(x, y) \, dA \]
\[ = \iint_R 4 + x^2 - y^2 \, dA \]
\[ = \int_{-1}^{1} \left[ \int_{0}^{2} 4 + x^2 - y^2 \, dy \right] \, dx \]
\[ = \int_{-1}^{1} \left( 4y + \frac{x^2y}{2} - \frac{y^3}{3} \bigg|_{y=0}^{y=2} \right) \, dx \]
\[ = \int_{-1}^{1} 8 + 2x^2 - \frac{8}{3} \, dx \]
\[ = \int_{-1}^{1} \frac{4}{3}x^3 + 2x^2 \, dx \]
\[ = \frac{4}{3}x^4 + \frac{2}{3}x^3 \bigg|_{-1}^{1} \]
\[ = 6 + 6 = 12 \]

Ex

Find \( \int_{-1}^{1} \int_{0}^{2} 5 \, dx \, dy \) by interpreting it as a volume.
Rectangular prism (box)

Dimensions: $1 \times 5 \times 5$

Volume: $25$