Method of substitution ("u-substitution")
Another method of finding antiderivatives.

Ex. \( \int \sqrt{2x-3} \, dx = ? \)

Try to replace this by an indet \( \int \) that's easier to do: introduce \( u = 2x-3 \)

Then \( \int \sqrt{2x-3} \, dx = \int \sqrt{u} \, dx \)

To deal with the "\( dx \)" part: \( \frac{du}{dx} = 2 \) i.e. \( du = 2 \, dx \)

\( \frac{1}{2} \, du = dx \)

Substitute that for \( dx \):

\[
\int \sqrt{u} \, dx = \int \sqrt{u} \left( \frac{1}{2} \, du \right) = \frac{1}{2} \int u^{\frac{1}{2}} \, du = \frac{1}{2} \cdot \frac{2}{3} \, u^{\frac{3}{2}} + C = \frac{1}{3} \, u^{\frac{3}{2}} + C = \frac{1}{3} \,(2x-3)^{\frac{3}{2}} + C
\]

(or: \( \frac{1}{3} \sqrt{(2x-3)^3} + C \))

Q 1. \( \int 7xe^{x^2} \, dx = ? \)

\( u = x^2 \)
\( du = 2x \, dx \)
\( \frac{du}{dx} = 2x \)

\( \int e^{-x^2} \, dx = ? \)
\[
\frac{1}{2} \int \frac{e^x}{x} \, dx = \int 7e^{x^2} \, dx \\
= \int 7e^u \cdot \frac{1}{2} \, du \\
= \frac{7}{2} \int e^u \, du = \frac{7}{2} e^u + C = \frac{7}{2} e^{x^2} + C
\]

For \( \int e^{-x^2} \, dx \): the antiderivative exists, but there's no formula for it in terms of \sin, \cos, \tan, \exp, \log, \ldots \)

Ex.
1) \( \int \sin^2 \theta \cos \theta \, d\theta = ? \)

Possibilities:
- \( u = \sin \theta \)
- \( u = \sin^2 \theta \)
- \( u = \sin^3 \theta \)
- \( u = \cos \theta \)

If take \( u = \sin \theta \), \( du = \cos \theta \, d\theta \)

Then \( \int \sin^2 \theta \cos \theta \, d\theta = \int u^2 \, du = \frac{1}{3} u^3 + C = \frac{1}{3} \sin^3 \theta + C \)

2) \( \int \frac{\cos (\ln t)}{t} \, dt = ? \)

\( u = \ln t \)
\( du = \frac{dt}{t} \)

\( \int \cos (\ln t) \frac{dt}{t} = \int \cos (u) \, du = \sin (u) + C = \sin(\ln t) + C \)

Q. \( \int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx \)

\( u = \sqrt{x} \)
\( du = \frac{1}{2x^{-\frac{1}{2}}} \, dx \)
\( \int \frac{dx}{\sqrt{x}} \)

So, \( 2du = \frac{dx}{\sqrt{x}} \)
\[
= \int e^u \cdot 2 \, du = 2e^u + C = 2e^{\sqrt{x}} + C
\]

Q 1) \[\int \sin(2x) \, dx = ?\]

\[u = 2x\]
\[du = 2 \, dx\]
\[
= \int \sin(u) \, \frac{du}{2}
\]
\[= \frac{-\cos(u)}{2} + C = \frac{-\cos(2x)}{2} + C
\]

2) \[\int_0^{\pi/2} \sin(2x) \, dx = ?\]

\[u = 2x\]
\[1, 2, \times \]

\[\text{NOT} \quad \int_0^{\pi/2} \sin(u) \, \frac{du}{2} !\]

\[
\int_{x=0}^{x=\pi/2} \sin(2x) \, dx = \int_{u=0}^{u=\pi} \sin(u) \, \frac{du}{2} = \frac{1}{2} \left( -\cos(u) \right) \bigg|_{u=0}^{u=\pi}
\]
\[= \frac{1}{2} \left( -(-1) - (-1) \right) = 1
\]

Ex \[\int_{\pi/3}^{\pi/2} \left( \cos(3x) e^{3\sin(3x)} \right) \, dx = ?\]

\[u = \sin 3x\]
\[du = 3 \cos 3x \, dx\]
\[
= \int_{x=\pi/3}^{x=\pi/2} e^u \, \frac{du}{3}
\]
\[= \int_{u=\sin(\pi/3)}^{u=\sin(\pi/2)} \frac{1}{3} e^u \, du
\]
\[x=\pi/2 \rightarrow u = \sin \left( \frac{3\pi}{2} \right) = -1
\]
\[x=\pi/3 \rightarrow u = \sin \left( 3 \cdot \frac{\pi}{3} \right) = 0
\]
\[ \frac{1}{3} e^u \bigg|_0^{-1} = \frac{1}{3} (e^1 - 1) \quad \text{or} \quad \frac{1}{3} e^{x^2} \bigg|_{x = \pi/3}^{x = \pi/2} = \frac{1}{3} (e^{-1}) \]

Could also do:

\[
\frac{u = 3x}{du = 3 \, dx} \quad \int (\cos 3x) \, e^{3x} \, dx = \int (\cos u) \, e^{3u} \, \frac{du}{3}
\]

then take:

\[ t = \sin u \]

\[
\text{Ex} \quad \int \frac{x^2 + 16x + 8}{\sqrt{x^2 + 1}} \, dx = ? \quad \text{could do either:} \quad \frac{u = \frac{x}{2} + 1}{u = \frac{\sqrt{x^2} + 1}{2}}
\]

if we take \( u = \frac{x}{2} + 1 \)

then \( du = \frac{1}{2} \, dx \)

\[
\int \frac{x^2 + 16x + 8}{\sqrt{u}} \, 2 \, du = 2 \int \frac{x^2}{\sqrt{u}} + \frac{16x}{\sqrt{u}} + \frac{8}{\sqrt{u}} \, du
\]

and since \( u = \frac{x}{2} + 1 \), \( x = 2u - 2 \)

so this is

\[
= 2 \int \frac{(2u-2)^2}{\sqrt{u}} + \frac{16(2u-2)}{\sqrt{u}} + \frac{8}{\sqrt{u}} \, du
\]

\[
= 2 \int \frac{4u^2 - 8u + 4}{\sqrt{u}} + \frac{32u - 32 + 8}{\sqrt{u}} \, du
\]

\[
= 2 \int \frac{4u^2 + 24u - 20}{\sqrt{u}} \, du
\]

\[
= 2 \int \frac{4u^{3/2} + 24u^{1/2} - 20u^{-1/2}}{\sqrt{u}} \, du
\]

\[
= \cdots
\]
\[ Q \]
\[
\int \frac{dx}{\sqrt{x}(1+x)} = \int \frac{du}{\sqrt{x} + x^{3/2}}
\]
\[
= \int \frac{2\,du}{1+u^2}
\]
\[
= 2 \tan^{-1} u + C
\]
\[
= 2 \tan^{-1} \sqrt{x} + C
\]