Imagine you finish a HW problem:

$$\int \text{(stuff)} = \ln \left| \frac{x}{6} + \frac{\sqrt{x^2 - 7x + 9}}{6} \right| + C$$

and QUEST shows $$\ln \left| x + \sqrt{x^2 - 7x + 9} \right| + C$$

This is actually correct!

Because

$$\ln \left| \frac{x}{6} + \frac{\sqrt{x^2 - 7x + 9}}{6} \right| + C$$

$$= \ln \left| x + \sqrt{x^2 - 7x + 9} \right| + \ln \left( \frac{1}{6} \right) + C$$

just a constant!

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Last time: improper integrals

$$\int_{a}^{\infty} f(x) \, dx$$

By definition,

$$\int_{a}^{\infty} f(x) \, dx = \lim_{t \to \infty} \int_{a}^{t} f(x) \, dx$$

Last time we did:

$$\int_{1}^{\infty} \frac{1}{x^2} \, dx = \lim_{t \to \infty} \left[ -\frac{1}{x} \right]_{1}^{t} = 1 \quad \text{convergent}$$

$$\int_{1}^{\infty} \frac{1}{x} \, dx = \lim_{t \to \infty} \ln t = \infty \quad \text{(DNE)}$$

$$\left( \frac{1}{x^2} \text{ gets } 0 \text{ “faster” than } \frac{1}{x} \text{ does, as } x \to \infty \right)$$
General rule: for \(a > 0\), \(\int_a^\infty \frac{1}{x^p} \, dx\) is **convergent** if \(p > 1\)

**divergent** if \(p \leq 1\)

\[ \int_{1042}^{\infty} \frac{1}{x^3} \, dx \text{ is convergent} \quad (3 > 1) \]

\[ \int^{\infty}_{\frac{1}{2}} \frac{1}{\sqrt{x}} \, dx \text{ is divergent} \quad \left( \frac{1}{2} < 1 \right) \]

\[ \quad \int_{-\infty}^{0} xe^x \, dx = ? \quad \text{(does it converge, and if so, what is the value?)} \]

We define this improper integral as

\[
\lim_{t \to -\infty} \left( \int_{-\infty}^{t} xe^x \, dx \right)
\]

**IBP:** \( u = x \quad v = e^x \)
\( du = dx \quad dv = e^x \, dx \)

\[
xe^x \bigg|_{-\infty}^{0} - \int_{-\infty}^{0} e^x \, dx
\]

\[= -te^t - (1 - e^t)\]

\[= -te^t - 1 + e^t\]

\[
\lim_{t \to -\infty} (-te^t - 1 + e^t)
\]

\[= \text{use L'Hopital's rule!} \quad -1 + 0
\]

\[
\lim_{t \to -\infty} e^t = 0
\]

\[e^{100} = \frac{1}{e^{100}} = \frac{1}{3^{100}} \text{ small} \]
\[ \lim_{t \to -\infty} \frac{-t}{e^t} \text{ looks like } \frac{\infty}{\infty} \]

\[ \lim_{t \to -\infty} \frac{-1}{-e^t} \to \frac{1}{\infty} = 0 \]

So finally \[ \lim_{t \to -\infty} (-te^t - 1 + e^t) = 0 - 1 + 0 = -1 \]

(Convergent)

Another kind of improper integral: \[ \int_a^b f(x) \, dx \] where \( f(x) \) becomes \( \infty \) for some \( x \) in \([a, b]\).

(i.e. \( f(x) \) has a vertical asymptote)

Here \( \int_a^b f(x) \, dx \) means

\[ \lim_{t\to b^-} \int_a^t f(x) \, dx \]

Similarly for

\[ \lim_{t\to a^+} \int_t^b f(x) \, dx \]

here \( \int_a^b f(x) \, dx \) means

\[ \int_a^b f(x) \, dx \]

Does \[ \int_2^5 \frac{1}{\sqrt{x-2}} \, dx \] converge? If so, to what?

\[ \int_{-2}^{3} \frac{1}{x} \, dx \]
\[
\lim_{t \to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}} \, dx
\]

\[
= \lim_{t \to 2^+} \left( 2\sqrt{x-2} \right|_t^5
\]

\[
= \lim_{t \to 2^+} (2\sqrt{3} - 2\sqrt{2-t})
\]

\[
= 2\sqrt{3} - 0
\]

\[
= 2\sqrt{3} \quad \text{(convergent)}
\]

A general rule: \( \int_0^a \frac{1}{x^p} \, dx \) is
\[
\begin{cases}
\text{convergent} & p < 1 \\
\text{divergent} & p \geq 1
\end{cases}
\]

\[
(\text{case for} \quad \int_0^a \frac{1}{c(x-c)^p} \, dx)
\]

\[
(\text{we just did example} \quad p = \frac{1}{2})
\]

For \( \int_{-2}^3 \frac{1}{x} \, dx \):

\[\text{vertical asymptote at } x = 0!\]

\[\text{We define it to be}\]

\[
\int_{-2}^0 \frac{1}{x} \, dx + \int_0^3 \frac{1}{x} \, dx
\]

\[\text{both divergent} \implies \text{this } \int \text{ is divergent.}\]

\[
\lim \int_{-2}^t + \int_{-t}^3
\]
\[ \int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx = \] 

(We define this integral by splitting it up:)

\[ \int_{0}^{\infty} \frac{1}{1+x^2} \, dx + \int_{-\infty}^{0} \frac{1}{1+x^2} \, dx \]

\[ = \lim_{t \to \infty} \int_{0}^{t} \frac{1}{1+x^2} \, dx + \lim_{t \to -\infty} \int_{t}^{0} \frac{1}{1+x^2} \, dx \]

\[ = \lim_{t \to \infty} (\tan^{-1} x \, |_{0}^{t}) + \lim_{t \to -\infty} (\tan^{-1} x \, |_{t}^{0}) \]

\[ = \lim_{t \to \infty} \tan^{-1} t - \lim_{t \to -\infty} \tan^{-1} t \]
\[
\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2} < \int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx = \pi
\]

\[ y = \tan x \quad \text{for} \quad \frac{-\pi}{2} < x < \frac{\pi}{2} \]

\[ \text{Ex.} \quad \int_{0}^{\infty} \cos x \, dx \]

\[ = \lim_{t \to \infty} \sin t \quad \text{DNE} \]

so this integral is divergent.

**Comparison Theorem**

Q. Does \( \int_{4}^{\infty} \frac{\sin^2(x)}{x^7} \, dx \) converge?
Yes, because \( \int_4^\infty \frac{1}{x^2} \, dx \) converges (p-test) and \( \alpha \frac{\sin x}{x^2} < \frac{1}{x^2} \).