Lecture 14

Exam 2 Tue Oct 31 7-9pm

Last time: improper integrals

So far, always looked at functions $f(x)$

In reality things often depend on >1 variable —
- e.g. heat index depends on temperature and humidity $f(x,y)$
- price of cheese depends on supply and demand
- productivity — supply of labor and supply of capital

Partial derivatives

If we have $f(x,y)$ we ask:

- How does $f(x,y)$ change when we vary $x$ and hold $y$ fixed?

Define $\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$

"partial deriv with respect to $x$"

To calculate $\frac{\partial f}{\partial x}$, treat $y$ as a constant and $x$ as a variable.

Similarly, to calculate $\frac{\partial f}{\partial y}$, treat $x$ as a constant and $y$ as a variable.

Q. If $f(x,y) = x^2 \sin(y)$ what is $\frac{\partial f}{\partial x}$? what is $\frac{\partial f}{\partial y}$?

What is $\frac{\partial f}{\partial x}$ at $(x,y) = (1, \frac{\pi}{2})$?

A. $f(x,y) = x^2 \sin(y)$
\[
\frac{df}{dx} = 2x \sin(y) \quad \frac{df}{dy} = x^2 \cos(y)
\]

Can also take second derivatives:
\[
\frac{\partial^2 f}{\partial x^2} = \frac{2}{\partial x} \left( \frac{\partial f}{\partial x} \right) \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{2}{\partial y} \left( \frac{\partial f}{\partial x} \right)
\]
\[
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right)
\]

Q. For \( f(xy) = 4x^2 y + 7 \sin(x) + 8 \)

What are all the partial derivatives, and 2nd partial derivatives?

\[
\frac{\partial f}{\partial x} = 8xy + 7 \cos(x)
\]

\[
\frac{\partial^2 f}{\partial x^2} = \frac{2}{\partial x} \left( 8xy + 7 \cos(x) \right) = 8y - 7 \sin(x)
\]

\[
\frac{\partial^2 f}{\partial y \partial x} = \frac{2}{\partial y} \left( 8xy + 7 \cos(x) \right) = 8x
\]

\[
\frac{\partial f}{\partial y} = 4x^2
\]

\[
\frac{\partial^2 f}{\partial x \partial y} = \frac{2}{\partial x} \left( 4x^2 \right) = 8x
\]

\[
\frac{\partial^2 f}{\partial y^2} = \frac{2}{\partial y} \left( 4x^2 \right) = 0
\]

Note: \( \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \)!

(Always true for functions which are nice enough — derivatives exist and are continuous)
Q. \[ \text{Say } f(xy) = \sin(xy). \]
\[ \text{Compute all the second derivatives.} \]

\[
\begin{align*}
 f_x &= y \cos(xy) \\
 f_y &= x \cos(xy)
\end{align*}
\]

\[
\begin{align*}
 f_{xx} &= y \cdot y \cdot (-\sin(xy)) \\
 f_{yy} &= x \cdot x \cdot (-\sin(xy)) \\
 &= -y^2 \sin(xy) \\
 &= -x^2 \sin(xy)
\end{align*}
\]

\[
\begin{align*}
 f_{xy} &= \frac{2}{dx} f_y = \frac{2}{dx} (x \cos(xy)) \\
 &= \cos(xy) + x \cdot y \cdot (-\sin(xy)) \\
 &= \cos(xy) - xy \sin(xy)
\end{align*}
\]

Check: \[ f_{yx} = \frac{2}{dy} f_x = \frac{2}{dy} (y \cos(xy)) \]

\[ = \cos(xy) - y \times \sin(xy) \]

---

**What's the meaning of partial derivatives?**

**If we have** \( f(xy) \)** we can make its graph**

\[ Z = f(xy) \]
e) say $f(x,y) = e^{-x^2-y^2}$. What do partial deriv. mean?

Q Calculate $f_x$ and $f_y$ and $f_x(1,0)$ $f_y(1,0)$

$f_x = -2x e^{-x^2-y^2}$
$f_y = -2y e^{-x^2-y^2}$

$f_x(1,0) = -2(1) e^{-1} = -2/e < 0$
$f_y(1,0) = -2(0) e^{-1} = 0$

$f_x(1,1) = -2 e^{-2} = -2/e^2 < 0$
$f_y(1,1) = -2 e^{-2} = -2/e^2 < 0$

Q $f_x(1,0) < 0$. What does this mean for the graph of $f$?
It means that $f$ is decreasing as we increase $x$, with $y$ fixed.

Q $f_y(1,0) = 0$. What does this mean for the graph of $f$?
Means that with $x$ fixed, $y=0$ is a critical point for $f(1,y)$.
(in fact, local max)

Contour plots

$f(x,y) = 10e^{-x^2-y^2}$

At $(x,y)$ $f_x < 0$
$f_y < 0$

At $(1,0)$ $f_x < 0$