Lecture 15

Exam 2  Tue Oct 31 7-9pm  Tesler A21A

Course Sec 7.2-14.3  ie everything from trig integrals → partial derivatives

Second part of HW4  → first part of HW5

Last time: partial derivatives

\[ \frac{df}{dx} = f_x \quad \frac{df}{dy} = f_y \]

\[ \frac{df}{dx^2} = f_{xx} \quad \frac{df}{dy^2} = f_{yy} \]

\[ \frac{df}{dx} \frac{df}{dy} = f_{xy} \]

\[ \frac{df}{dy} \frac{df}{dx} = f_{yx} \]

These two are equal!

Q. At (x,y) is \( f_x \) positive, negative or zero?  \( f_x > 0 \)

At (x,y) is \( f_y \) positive, negative or zero?  \( f_y < 0 \)
How about integrating a function of two variables?

Q: What is the total volume under this graph (i.e., volume between graph \( z = f(x,y) \) and the xy-plane \( z = 0 \))?

Cut by planes at fixed \( y \):

\[ V = \int_0^2 A(y) \, dy \]

\[ A(y) = \int_0^3 f(x,y) \, dx \] gives cross section area

So

\[ V = \int_0^2 \left[ \int_0^3 f(x,y) \, dx \right] \, dy \]

Q: Suppose \( f(x,y) = 4xy + 3x^2 \). What's \( V \)?

\[ V = \int_0^2 \left[ \int_0^3 4xy + 3x^2 \, dx \right] \, dy \]

\[ = \int_0^2 \left[ \frac{2x^2 y + x^3}{3} \bigg|_{x=3} \right] \, dy \]

\[ = \int_0^2 \left[ 18y + 27 - 0 \right] \, dy \]
\[ \int_0^1 \left( \int_0^2 4xy + 3x^2 \, dy \right) \, dx \\
= \int_0^3 \left[ 2xy^2 + 3x^2 y \bigg|_0^2 \right] \, dx \\
= \int_0^3 (8x + 6x^2) \, dx \\
= 4x^2 + 2x^3 \bigg|_0^3 \\
= 36 + 54 = 90 \\
\]

Re: We could also do it in the other order:

Either order of integration gives same answer ("Fubini's Thm")

\[
\int_0^b \left( \int_c^d f(x,y) \, dy \right) \, dx = \int_c^d \left( \int_a^b f(x,y) \, dx \right) \, dy
\]

We also write this as

\[
\iint_R f(x,y) \, dA \\
R = \left\{0 \leq x \leq b, \quad c \leq y \leq d \right\}
\]

\[
dA = dx \, dy
\]

\[
\frac{dy}{dx}
\]
Q: If \( R = \{1 \leq x \leq 2, \ 0 \leq y \leq \pi\} \) and \( f(x,y) = y \sin(xy) \), what is \( \iint_R f(x,y) \, dA \)?

\[
\begin{align*}
\int_0^\pi \left( \int_1^2 y \sin(xy) \, dx \right) \, dy &= \int_0^\pi \left( \frac{-y \cos(xy)}{y} \right)_{x=2}^{x=1} \, dy \\
&= \int_0^\pi \left( -\cos(2y) + \cos(y) \right) \, dy \\
&= \left. \left( -\frac{\sin(2y)}{2} + \sin(y) \right) \right|_0^\pi \\
&= 2
\end{align*}
\]

\[\text{OR: could do } \int_1^2 \left( \int_0^\pi y \sin(xy) \, dy \right) \, dx \]

\(\text{do by IBP} \)
Q. Find the volume of the solid which lies under the graph of

\[ z = f(x, y) = 4 + x^2 - y^2 \]

and over the rectangle \( R = \{ -1 \leq x \leq 1, 0 \leq y \leq 2 \} \).

\[ f(x, y) \geq 0 \]

whenever \((x, y) \in R\).

So, \( V = \int_{-1}^{1} \left[ \int_{0}^{2} 4 + x^2 - y^2 \, dy \right] \, dx \)

\[ = \int_{-1}^{1} \left( 4y + x^2 y - \frac{1}{3} y^3 \Big|_{y=0}^{y=2} \right) \, dx \]

\[ = \int_{-1}^{1} \left( 8 + 2x^2 - \frac{8}{3} \right) \, dx \]

\[ = \int_{-1}^{1} \left( \frac{16}{3} + 2x^2 \right) \, dx \]

\[ = \left[ \frac{16}{3} x + \frac{2}{3} x^3 \right]_{-1}^{1} \]

\[ = \left( \frac{16}{3} + \frac{2}{3} \right) - \left( -\frac{16}{3} + \frac{2}{3} \right) \]

\[ = \frac{32}{3} = 12 \]
Find \( \int_{-1}^{4} \int_{2}^{3} 5 \, dx \, dy \) by interpreting it as a volume.