Welcome! M408M, Fall 2014. Multivariate Calculus.

Course web page: www.ma.utexas.edu/users/neitzke/teaching/408M
- contains first day handout, will contain lecture slides posted after each class

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TA: Dan Kubala dkubala@math.utexas.edu

Lectures Tu Th 11:00-12:30, CPE 2.214 (11:00-12:15 ± 2mn)

Discussion MW 8:00-9:00, UTC 1.116
or MW 2:00-3:00, CPE 2.220

Office hours Instructor: MW 11:00-12:00, RLM 9.134
TA: M 3:00-4:30, RLM 12.146
F 1:00-2:30, RLM 12.146

Homework via QUEST at quest.cns.utexas.edu
Due at 3am each Tue morning — can submit as you go
Lowest 2 dropped from grade
Working together strongly recommended!

2 midterm exams (in class)
Oct 2, Nov 4
Can replace lowest midterm grade with final grade

Final exam

Many resources available:
Office hours
Lecture slides
Textbook
Calc Lab (M-F 2-7, Painter 5.42)
Sanger Center (Tester A115)
Your fellow students!
Parameterized Curves (Ch 10.1)

Until now, you’ve mostly considered

Graph of the function:

\[ y = f(x) \]

\[ (x, f(x)) \]

\[ x \]

\[ y \]

\[ \text{Example: } f(x) = 1 - x^2 \]

\[
\begin{array}{c|c}
  x & y \\
  \hline
  0 & 1 \\
  1 & 0 \\
  -1 & 0 \\
  2 & -3 \\
  -2 & -3 \\
\end{array}
\]

\[ y = 1 - x^2 \]

\[ (-1, 0) \]

\[ (0, 1) \]

\[ (1, 0) \]

\[ (-2, -3) \]

\[ (2, -3) \]

But, we can’t represent every curve this way!

(Vertical line test)

Instead, try writing:

\[ x = f(t) \]

\[ y = g(t) \]

\[
\begin{array}{c|c|c}
  t & x & y \\
  \hline
  2 & 1 & \frac{5}{3} \\
  1 & 0 & \frac{3}{1} \\
  -1 & -2 & -1 \\
  -2 & 1 & -3 \\
\end{array}
\]

Ex: Say we put

\[ x = t^2 - 3 \]

\[ y = 2t + 1 \]
Looks like parabola! To see it is parabola, eliminate $t$:

\[
y = 2t + 1 \\
t = \frac{1}{2} (y - 1) \\
x = \frac{1}{4} y^2 - y - \frac{11}{4}
\]

Q: Say $y = f(x)$

\[
y = \frac{1}{\sin(x^2 - 4)} + \tan(x)
\]

Can this be written in parametric form?

Yes: take $x = t$

\[
y = \frac{1}{\sin(t^2 - 4)} + \tan(t)
\]

Ex. (Most famous example)

\[
\begin{array}{c|c|c|c}
\hline
x & y \\
\hline
0 & 0 & 1 \\
\frac{\pi}{4} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\
\frac{\pi}{2} & 1 & 0 \\
\pi & 0 & -1 \\
\frac{3\pi}{2} & -1 & 0 \\
2\pi & 0 & 1 \\
\hline
\end{array}
\]

$x^2 + y^2 = \sin^2 t + \cos^2 t = 1$

If we restrict $t$ to $0 \leq t \leq \pi$, then get half of a circle.
\[ x = \sin(3t) \]
\[ y = \cos(3t) \]

\[ x^2 + y^2 = \sin^2(3t) + \cos^2(3t) = 1 \quad \text{so still unit circle} \]

but we "go around 3 x faster"

---

\[ x = 2 \sin t \]
\[ y = \cos t \]

<table>
<thead>
<tr>
<th>( t )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \pi ) (</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>( \pi ) (</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>( 3\pi ) (</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2\pi</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ x^2 + y^2 = 4 \sin^2 t + \cos^2 t \]

\[ \frac{x^2}{4} + y^2 = 1 \quad \text{eq. of ellipse} \]

\[ \sqrt{2} \text{ minor axis} \]
\[ 2 \text{ major axis} \]

\[ \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ a = 2, b = 1 \right) \]
**Ex** \[x = \sin t\]
\[y = \sin t\]

<table>
<thead>
<tr>
<th>t</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\frac{\pi}{4})</td>
<td>(\sqrt{2}/2)</td>
<td>(\sqrt{2}/2)</td>
</tr>
<tr>
<td>(\frac{\pi}{2})</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(\pi)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\frac{3\pi}{2})</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>(2\pi)</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\[y = x\] but not every point on the line is included: only those with \(-1 \leq x \leq 1\)

Oscillate back and forth between \((-1, -1)\) and \((1, 1)\).

**Ex** Sketch the curve given by \[x = x(t)\] \[y = y(t)\]

**Answer:**

**Ex** Try \[x = \sin 2t\] \[y = \sin t\]!
Calculus with Parameterized Curves (Ch 10.2)

Finding tangents

Recall:

\[ y = f(x) \]

slope = \( \frac{dy}{dx} = \frac{d}{dx} f(x) \)

If we have a parameterized curve:

\[ x = f(t) \quad y = g(t) \]

What is the slope of the tangent line?

slope = \( \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \) as long as this is not \( 0/0 \)!

Ex. Say \( x(t) = 3t^2 + 1 \quad y(t) = \frac{2}{t} \)

Find slope of tangent line to this curve at \((x, y) = (4, 7)\).

\[
\begin{align*}
dy/dt &= -\frac{7}{t^2} \\
dx/dt &= 6t \\
\text{and we evaluate at } t = 1: \\
dy/dt &= -7 \\
dx/dt &= 6
\end{align*}
\]

\[ \text{slope} = \frac{dy/dt}{dx/dt} = \frac{-7}{6} \]

Rk. For a more complicated parameterized curve, knowing \((x, y)\) might not determine \(t\)!
Why does the formula \( \text{slope} = \frac{dy/dt}{dx/dt} \) work?

Mnemonic: "cancel the dt's" to get \( dy/dx \)

Imagine moving from \( t \) to \( t + \Delta t \), for \( \Delta t \) very small:

\[
\frac{\Delta y}{\Delta x} \approx \frac{dy/dt}{dx/dt} \approx \frac{dy/dt \cdot \Delta t}{dx/dt \cdot \Delta t} = \frac{dy/dt}{dx/dt}
\]

**Ex. Cycloid**

\[
x = t - \sin t \\
y = 1 - \cos t
\]

a) Find slope of tangent line at \( t = \frac{\pi}{3} \)

\[
\text{slope} = \frac{dy/dt}{dx/dt} = \frac{\sin t}{1 - \cos t} = \frac{\sqrt{3}/2}{1 - \frac{1}{2}} = \sqrt{3}
\]

b) Find the values of \( t \) when cycloid has horizontal or vertical tangents.

**Horizontal tangents:** slope = 0 \( \Leftrightarrow \) \( \frac{dy}{dt} = 0 \) and \( \frac{dx}{dt} \neq 0 \)

\( \sin t = 0 \) \( \Leftrightarrow t = (2n+1)\pi \) in any integer

**Vertical tangents:** slope = \( \infty \) \( \Leftrightarrow \) \( \frac{dx}{dt} = 0 \) and \( \frac{dy}{dt} \neq 0 \)

\( \cos t = 1 \) \( \Leftrightarrow \) \( t = 0 \) \( \sin t \neq 0 \) this never happens
But, we should also consider what happens at places where both \( \frac{dx}{dt} = 0 \) and \( \frac{dy}{dt} = 0 \). This happens at \( t = 2n\pi \), \( n \) any integer.

In this case, the slope formula \( \frac{dy/dt}{dx/dt} \) gives \( \frac{0}{0} \).

We get the correct slope in this case by L’Hospital Rule:

\[
\lim_{t \to 0} \frac{\sin t}{1-\cos t} = \lim_{t \to 0} \frac{\cos t}{\sin t} = +\infty
\]

Thus we do get vertical tangents at these points!

So altogether: horizontal tangents at \( t = (2n+1)\pi \)
vertical tangents at \( t = 2n\pi \)