Last time: parameterized curves \( x(t) \), \( y(t) \)

Slope of tangent line:
\[
\text{slope} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{as long as this isn't } \frac{0}{0}!
\]

(If it is \( \frac{0}{0} \), say at \( t=a \), then try \( \text{slope}(t=a) = \lim_{t \to a} \frac{dy/dt}{dx/dt} \))

Remark: Suppose we want to know the concavity of a parameterized curve — then want to compute "\( \frac{d^2y}{dx^2} \)". This is not given by \( \frac{d^2y/dt^2}{d^2x/dt^2} \)

Rather, we want \( \frac{d}{dx}(\text{slope}) \)

or equivalently \( \frac{d}{dt}(\text{slope}) = \frac{d^2y}{dx^2} \)

[why? rough idea:
we want \( \frac{d}{dx}(\text{slope}) \)
so first compute \( \frac{d}{dt}(\text{slope}) \)
then take \( \frac{d}{dx} \cdot \frac{d}{dt}(\text{slope}) \)]

Ex: Say have param. curve \( x=t^2 \), \( y=t^3-3t \)

Is this curve concave up or concave down at \( t=1 \)?

\[
\text{slope} = \frac{dy/dt}{dx/dt} = \frac{3t^2-3}{2t} = 0 \text{ at } t=1 \quad \text{(hence tangent)}
\]

"\( \frac{d^2y}{dx^2} \)"
\[
= \frac{d}{dx}(\text{slope}) = \frac{d}{dt}(\frac{3t^2-3}{2t}) \frac{1}{2t} = \cdots = \frac{3(1+t^2)}{4t^3}
\]
So at \( t=1 \), \( \frac{d^2 y}{dx^2} = \frac{b}{4} = \frac{3}{2} > 0 \Rightarrow \text{slope is increasing as we move to the right}\)

**Areas under curves** (Ch 10.2)

Recall:

\[
A = \int_a^b y(x) \, dx
\]

why? Approximate A by a sum over small rectangles:
- each has area \( y(x) \cdot \Delta x \)
- \( A \approx \text{sum of } y(x) \cdot \Delta x \text{ over all rect.} \)
- As \( \Delta x \to 0 \), this becomes \( A = \int_a^b y(x) \, dx \)

Now say have parameterized curve:

\( x = x(t) \)
\( y = y(t) \)

\[
A = \int \text{area } = \int_\alpha^\beta y(t) \frac{dx}{dt} \, dt
\]

[Why? Essentially same reason:
- \( y(t) \)
- \( \Delta x \approx \frac{dx}{dt} \Delta t \)]

For area of region between the curve and the y-axis, would use instead

\[
\int x \, dy
\]

\[
\int x \frac{dy}{dt} \, dt
\]
Ex. Find the area of the region between the x-axis and the parameterized curve

\[ x = 1 + e^t \]
\[ y = t - t^2 \]

When \( y = 0 \), i.e. \( t - t^2 = 0 \)
\[ \Rightarrow t(t - 1) = 0 \quad t = 0, t = 1 \]

At \( t = 0 \), \((x, y) = (2, 0)\)
At \( t = 1 \), \((x, y) = (1 + e, 0)\)

\[ A = \int_0^1 y \cdot \frac{dx}{dt} \, dt = \int_0^1 (t - t^2) e^t \, dt = \ldots = 3 - e \]

Use integration by parts

Ex. Find the area of a unit \( \frac{1}{4} \)-circle, using the description of the circle as a parameterized curve.

\[ x = \sin \theta \]
\[ y = \cos \theta \]

\[ 0 \leq \theta \leq \frac{\pi}{2} \]

\[ A = \int_0^{\frac{\pi}{2}} y \cdot \frac{dx}{d\theta} \, d\theta \]
\[ = \int_0^{\frac{\pi}{2}} \cos \theta \cos \theta \, d\theta \]
\[ = \int_0^{\frac{\pi}{2}} \cos^2 \theta \, d\theta \]
\[
\int_0^{\pi/2} \frac{1}{2} (1 + \cos 2\theta) \, d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4}\bigg|_0^{\pi/2} = \left(\frac{\pi}{4} + 0\right) - \left(0 + 0\right) = \frac{\pi}{4}
\]

**Arc length**

\[x(t), \ y(t)\]

Length:

\[L = \int_\alpha^\beta \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \, dt\]

(\(\alpha < \beta\))

**Why?** Approximate path by a polygonal path

Length of each segment:

\[
\Delta x \approx \frac{dx}{dt} \cdot \Delta t \quad \Delta y \approx \frac{dy}{dt} \cdot \Delta t
\]

\[
\text{length of segment} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \cdot \Delta t = \int_{\alpha}^{\beta} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \cdot dt
\]

sum up all the segments, take \(\Delta t \to 0\):

\[L = \int_{\alpha}^{\beta} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \, dt\]

**Ex:** Cycloid: \(x = \Theta - \sin \Theta\)

\[y = 1 - \cos \Theta\]

Length of one arch:

\[\Theta = 0 \quad \Theta = 2\pi\]
\[
L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta = \int_0^{2\pi} \sqrt{(1-\cos \theta)^2 + (\sin \theta)^2} \, d\theta \\
= \int_0^{2\pi} \sqrt{1-2\cos \theta + \cos^2 \theta + \sin^2 \theta} \, d\theta \\
= \int_0^{2\pi} \sqrt{2-2\cos \theta} \, d\theta \\
= \sqrt{2} \int_0^{2\pi} \sqrt{1-\cos \theta} \, d\theta \\
= \sqrt{2} \int_0^{2\pi} \sqrt{2 \sin^2 \frac{\theta}{2}} \, d\theta \\
= 2 \int_0^{2\pi} |\sin \frac{\theta}{2}| \, d\theta \\
= 2 \int_0^{2\pi} \sin \frac{\theta}{2} \, d\theta \quad \text{[since } \sin \frac{\theta}{2} > 0 \text{ for } 0 \leq \theta \leq 2\pi]\]
\[
= 2(-2 \cos \frac{\theta}{2}) \bigg|_0^{2\pi} = 2(2+2) = 8
\]

**Surface area for surfaces of revolution**

What is the surface area of the surface obtained by rotating this curve around the x-axis?

**S** = \[\int_0^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt\]

\[\text{[Why? Cut into cylindrical "ribbons" with surface area } = 2\pi y \cdot \sqrt{(\Delta x)^2 + (\Delta y)^2} \text{]}\]
Exercise: Take the curve

\[ x = a \cdot \cos^3 \theta \]
\[ y = a \cdot \sin^3 \theta \]

(a > 0)

\[ 0 \leq \theta \leq \frac{\pi}{2} \]

Rotate around x-axis:

\[ S = \int_0^{\pi/2} 2\pi y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta \]

\[ = \int_0^{\pi/2} 2\pi \left( a \sin^3 \theta \right) \sqrt{(-3a \sin \theta \cos^2 \theta)^2 + (3a \cos \theta \sin^2 \theta)^2} \, d\theta \]

\[ = \int_0^{\pi/2} 2\pi a \sin^3 \theta \sqrt{9a^2 \sin^2 \theta \cos^2 \theta + 9a^2 \cos^2 \theta \sin^2 \theta} \, d\theta \]

\[ = \int_0^{\pi/2} 6\pi a^2 \sin^3 \theta \sqrt{\sin^2 \theta \cos^2 \theta (\sin^2 \theta + \cos^2 \theta)} \, d\theta \]

\[ = \int_0^{\pi/2} 6\pi a^2 \sin^3 \theta \sin \theta \cos \theta \, d\theta \]

\[ = \int_0^{\pi/2} 6\pi a^2 \sin^4 \theta \cos \theta \, d\theta \]

\[ = 6\pi a^2 \left( \frac{\sin^5 \theta}{5} \right) \bigg|_0^{\pi/2} = \frac{6\pi a^2}{5} (1 - 0) = \frac{6\pi a^2}{5} \]