Reminder: I will have office hour today 1:00-2:00 pm
I will have no office hour tomorrow
Dan will have office hour tomorrow (Wed) 3:00-4:30 pm

Exam 1 Thursday You only need pencils + erasers
(Calculators are not permitted)
Grades should be posted Monday evening

**Vector functions (Ch 13.1 cont'd)**

\[ \vec{r}(t) = \langle f(t), g(t), h(t) \rangle \]

Recall: vector functions are closely related to parameterized curves — if put the tails of all vectors \( \vec{r}(t) \) at the origin, the tips of \( \vec{r}(t) \) sweep out a curve.
From now on, sloppily say vector \( \vec{r} \) = parameterized curve.

**Ex** Write a parameterization for the intersection between (in 3 dim)
the locus \( x^2 + y^2 = 1 \) cylinder
and \( y + z = 2 \) plane

Expect this intersection will look like an ellipse.
Idea: $z = 2 - y$ so in particular, $z$ is totally determined by $y$.

So, let's deal with $x$ and $y$ first, then come back to $z$.

To parameterize $x$ and $y$, say $x = \cos t \quad y = \sin t$

then using $z = 2 - y$ have $z = 2 - \sin t$

So altogether $\vec{r} = \langle x(t), y(t), z(t) \rangle = \langle \cos t, \sin t, 2 - \sin t \rangle$

Calculus of Vector Functions, cont'd (Ch 13.2)

Last time: differentiation of vector functions

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle \quad \vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$

Interpretation: $\vec{r}'(t)$ is tangent vector to the curve $\vec{r}(t)$

Ex. Find the tangent line to the curve $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at $t = 1$.

(in parametric form)

Need a point on the tangent line and a vector parallel along the tangent line.

Point on the line: $\vec{r}(1) = \langle 1, 1, 1 \rangle$

Vector along the line: $\vec{r}'(t) = \langle 1, 2t, 3t^2 \rangle$

so $\vec{r}'(1) = \langle 1, 2, 3 \rangle$

So the tangent line is $\vec{l}(t) = \vec{r}(1) + t \vec{r}'(1)$
\[
\begin{align*}
\langle 1,1,1 \rangle + t \langle 1,2,3 \rangle &= \langle 1+t, 1+2t, 1+3t \rangle \\
\text{Velocity} \quad &
\text{If we have a path } P(t) \text{ with displacement from the origin given by a vector } \vec{r}(t) \text{ then } \vec{r}'(t) = \vec{v}(t) \text{ is the velocity.} \\
\text{The length } \|\vec{v}(t)\| \text{ is the speed.} \\
(\text{These definitions apply either in 2-d or in 3-d})
\end{align*}
\]

\[
\begin{align*}
\text{Ex: } &\quad \text{If } \vec{r}(t) = \langle t, t^2+1 \rangle \\
&\quad \text{then } \vec{r}'(t) = \langle 1, 2t \rangle \\
&\quad \text{e.g. if } t=0, \vec{r}'(0) = \langle 1, 0 \rangle \\
&\quad \vec{r}(0) = \langle 0, 1 \rangle \\
&\quad t=1, \vec{r}'(1) = \langle 1, 2 \rangle \\
&\quad t=-1, \vec{r}'(-1) = \langle 1, -2 \rangle \\
\text{Speed: } &\quad \|\vec{r}'(t)\| = \|\langle 1, 2t \rangle\| = \sqrt{1+4t^2}
\end{align*}
\]

**Differentiation rules**

\[
\begin{align*}
\frac{d}{dt} (\vec{u}(t) + \vec{v}(t)) &= \vec{u}'(t) + \vec{v}'(t) \\
\frac{d}{dt} (c \cdot \vec{u}(t)) &= c \cdot \frac{d}{dt} \vec{u}(t) \\
\frac{d}{dt} (f(t) \vec{u}(t)) &= f'(t) \vec{u}(t) + f(t) \vec{u}'(t) \\
\frac{d}{dt} (\vec{u}(t) \cdot \vec{v}(t)) &= \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t) \\
\frac{d}{dt} (\vec{u}(t) \times \vec{v}(t)) &= \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)
\end{align*}
\]

**careful with order!**
\[
\frac{d}{dt} \left( \vec{u}(f(t)) \right) = f'(t) \vec{u}'(f(t)) \quad \text{[Chain Rule for vector functions]}
\]

Proofs of these in text: by reducing to the rules for ordinary functions.

Ex If \( \vec{u}(t) = \vec{a} + t \vec{b} + t^2 \vec{c} \)

then \( \vec{u}'(t) = \vec{b} + 2t \vec{c} \)

"just as if \( \vec{a}, \vec{b}, \vec{c} \) were constant #s"

Ex Prove: If \( \vec{r}(t) \) lies (for all \( t \)) on a sphere of radius \( c \) around the origin, then \( \vec{r}(t) \perp \vec{r}'(t) \).

We know that

\[
\sqrt{x(t)^2 + y(t)^2 + z(t)^2} = c
\]

distance from \( (x,y,z) \) to \( (0,0,0) \)

i.e. \[
\|\vec{r}(t)\| = c
\]
\[
\|\vec{r}(t)\|^2 = c^2
\]
\[
\vec{r}(t) \cdot \vec{r}(t) = c^2
\]

Take \( \frac{d}{dt} \) of both sides: \[
\frac{d}{dt} \left( \vec{r}(t) \cdot \vec{r}(t) \right) = \frac{d}{dt} (c^2)
\]
\[
\vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t) = 0
\]
\[
2 \vec{r}'(t) \cdot \vec{r}(t) = 0
\]
\[
\vec{r}'(t) \cdot \vec{r}(t) = 0 \quad \therefore \quad \vec{r}'(t) \perp \vec{r}(t).
\]
Integrals of vector functions

We define the integral of a vector function much like we did for scalar functions:

by Riemann sums (not as area under something!)

\[ \int_a^b \vec{\tau}(t) \, dt = \lim_{n \to \infty} \sum_{i=1}^{n} \vec{\tau}(t_i^*) \Delta t \]

\[ \Delta t = \frac{b-a}{n} \]

If \( \vec{\tau}(t) = \langle f(t), g(t), h(t) \rangle \)

then \( \int_a^b \vec{\tau}(t) \, dt = \langle \int_a^b f(t) \, dt, \int_a^b g(t) \, dt, \int_a^b h(t) \, dt \rangle \)

Key fact: Fundamental Theorem of Calculus for vector functions

If \( \vec{R}'(t) = \vec{\tau}(t) \) then \( \int_a^b \vec{\tau}(t) \, dt = \vec{R}(b) - \vec{R}(a) \).

Therefore, if \( \vec{\tau}(t) = \langle 1, e^t, t^2 \rangle \)

then what is \( \int_0^2 \vec{\tau}(t) \, dt \) ?

\( \vec{\tau}(t) = \vec{R}'(t) \) when

\( \vec{R}(t) = \langle t, e^t, \frac{1}{3}t^3 \rangle + \vec{C} \)

\( \int_0^2 \vec{\tau}(t) \, dt = \vec{R}(2) - \vec{R}(0) = (\langle 2, e^2, \frac{8}{3} \rangle + \vec{C}) - (\langle 0, 1, 0 \rangle + \vec{C}) \)

(= \( \vec{R}'(t) \bigg|_0^2 \) = \( \langle 2, e^2, 1, \frac{8}{3} \rangle \))

(We'd get the same answer by \( \int_0^2 \) each component separately)
Indefinite integrals of vector functions:

given \( \vec{r}(t) \), \( \int \vec{r}(t) \, dt \) means any antiderivative of \( \vec{r}(t) \).

Ex.
\[
\int (\cos t) \vec{i} + (\sin t) \vec{j} + t \vec{k} \, dt
\]
\[= (\sin t) \vec{i} + (-\cos t) \vec{j} + \frac{1}{2} t^2 \vec{k} + \vec{C}\]

Ex. Suppose a particle moves with velocity
\[
\vec{v}(t) = \langle t, e^t, te^t \rangle
\]
and at \( t = 0 \) the particle is at \( (1,1,1) \).

Find the position \( \vec{r}(t) \).

\[
\vec{v}(t) = \vec{r}'(t) \quad \text{i.e.} \quad \vec{r}(t) = \int \vec{v}(t) \, dt
\]
\[= \int \langle t, e^t, te^t \rangle \, dt
\]
\[= \left\langle \frac{1}{2} t^2, e^t, (t-1)e^t \right\rangle + \vec{C}\]

\[
\vec{r}(0) = \langle 0, 1, -1 \rangle + \vec{C} \quad \text{and we know} \quad \vec{r}(0) = \langle 1, 1, 1 \rangle
\]

5. \( \vec{C} + \langle 0, 1, -1 \rangle = \langle 1, 1, 1 \rangle \)
\[\vec{C} = \langle 1, 1, 1 \rangle - \langle 0, 1, -1 \rangle = \langle 1, 0, 2 \rangle\]
\[ F(t) = \left< \frac{1}{2} t^2, e^t, (t-1)e^t \right> + \left< 1, 0, 2 \right> \\
= \left< \frac{1}{2} t^2 + 1, e^t, (t-1)e^t + 2 \right> \]