Midterm exam 2: Tuesday Nov 4, 11:00 - 12:15
15 questions
14 (format the same as midterm 1)

Cover material through Lecture 17 (today)

HW10 due Thursday Nov 6 3am
HW11 due Tuesday Nov 11 3am

Last time: tangent planes to graphs
\[ z = f(x, y) \]

at \( (x_0, y_0, f(x_0, y_0)) \):
\[ z - z_0 = (x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0) \]

linear approximation:
\[ \Delta z \approx \Delta x \cdot f_x(x_0, y_0) + \Delta y \cdot f_y(x_0, y_0) \]

Chain rule for multiple variables (Ch 14.5)

Recall chain rule for functions of one variable:
Say

- \( y \) is a function of \( x \)
- \( x \) is a function of \( t \)

Then \( y \) is a function of \( t \):
\( y = f(g(t)) \)

\( y \) depends on \( t \) implicitly, through \( x \)

Chain rule:
\[ \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = f'(g(t)) \cdot g'(t) \]
With more than one variable, the way variables depend on each other can be more complicated.

Let’s start with:

- \( z = f(x, y) \)
- \( x, y \) are both functions of \( t \):
  \[ x = g(t) \]
  \[ y = h(t) \]

- Thus \( z \) is a function of \( t \) through \( x \) and \( y \),
  \[ z = f(g(t), h(t)) \]

Q: What is \( \frac{dz}{dt} \)?

A: given by multivariable chain rule,

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}
\]

Ex: \( S o l \) \( z = z(x, y) = x^2 + y^2 + xy \)

\( x = x(t) = \sin t \)
\( y = y(t) = e^t \)

\( z \) is implicitly a function of \( t \).
Let’s compute \( \frac{dz}{dt} \).

\[
\frac{\partial z}{\partial x} = 2x + y \quad \frac{\partial z}{\partial y} = 2y + x \quad \frac{dx}{dt} = \cos t \quad \frac{dy}{dt} = e^t
\]

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}
= (2x + y) \cos t + (2y + x) e^t
= (2\sin t + e^t) \cos t + (2e^t + \sin t) e^t
= 2\sin t \cos t + e^t \cos t + 2e^{2t} + e^t \sin t
\]
What is the meaning of this?

\[ z(x, y) = x^2 + y^2 + xy \]

and we study what happens to the line \( z \) along the parameterized curve:

\[ x = x(t) = \sin t \]
\[ y = y(t) = e^t \]

We could also have substituted directly:

\[ z = \sin^2 t + e^{2t} + e^t \sin t \]
\[ \frac{dz}{dt} = 2 \sin t \cos t + 2e^{2t} + e^t \cos t + e^t \sin t \]

\[ \checkmark \text{ matches what we got from the multi-variable chain rule.} \]

Ex

Volume of cylinder \( V = \pi r^2 h \)

Suppose \( r \) is increasing at the rate 3 cm/s
\( h \) is decreasing at the rate 1 cm/s

What is \( \frac{dV}{dt} \) when \( r = 10 \text{ cm}, h = 20 \text{ cm} \)?

Chain rule:

\[ \frac{dV}{dt} = \frac{\partial V}{\partial r} \cdot \frac{dr}{dt} + \frac{\partial V}{\partial h} \cdot \frac{dh}{dt} \]
\[ = (2\pi r h) \cdot \frac{dr}{dt} + (\pi r^2) \cdot \frac{dh}{dt} \]
\[ = 2\pi (10 \text{ cm})(20 \text{ cm})(3 \text{ cm/s}) + \pi (10 \text{ cm})^2 (-1 \text{ cm/s}) \]
\[ = 1200\pi \text{ cm}^3/\text{s} - 100\pi \text{ cm}^3/\text{s} = 1100\pi \text{ cm}^3/\text{s} \]
Why is the multivariable chain rule true?
To prove it, use linear approximation:

\[ z = z(x, y) \]
\[ x = x(t) \]
\[ y = y(t) \]

We vary \( t \) by an amount \( \Delta t \), and to determine the change \( \Delta z \)

\[ \Delta z \approx \frac{\partial z}{\partial x} \cdot \Delta x + \frac{\partial z}{\partial y} \cdot \Delta y \]

\[ \frac{\Delta z}{\Delta t} \approx \frac{\partial z}{\partial x} \cdot \frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y} \cdot \frac{\Delta y}{\Delta t} \]

In the limit \( \Delta t \to 0 \) this becomes

\[ \frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \]

as we wanted.

**Warning:** the mnemonic in 1 variable \( \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \)

doesn't extend well to 2 variables:

\[ \frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = 2 \cdot \frac{dz}{dt} \]

??? wrong!

A mnemonic that does work: take the total differential

\[ dz = \frac{\partial z}{\partial x} \, dx + \frac{\partial z}{\partial y} \, dy \]

and divide by \( dt \) on both sides.

Another situation: \( z = f(x, y) \)

\[ x = g(s, t) \]
\[ y = h(s, t) \]

Then \( z = f(g(s, t), h(s, t)) \) is indirectly a function of \( s \) and \( t \).
So, can ask for \( \frac{\partial z}{\partial s}, \frac{\partial z}{\partial t} \).

e.g. for \( \frac{\partial z}{\partial s} \), do just as we did above, treating \( t \) as a constant.

So \[ \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \]

\[
\begin{bmatrix}
\text{holding } t \text{ fixed} & \text{holding } y \text{ fixed} & \text{holding } x \text{ fixed} & \text{holding } t \text{ fixed}
\end{bmatrix}
\]

Similarly \[ \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \]

**Ex.**

Say \( z(x,y) = x^2 y^3 \)

\[ x(s,t) = s \cos t \]
\[ y(s,t) = s \sin t \]

Find \( \frac{\partial z}{\partial s} \) and \( \frac{\partial z}{\partial t} \) at \( (s,t) = (1, \frac{\pi}{4}) \).

At \( (s,t) = (1, \frac{\pi}{4}) \) have \( x = \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}}, z = \frac{1}{4\sqrt{2}} \)

\[
\begin{align*}
\frac{\partial z}{\partial x} &= 2xy^3 = \frac{3}{4} \\
\frac{\partial z}{\partial y} &= 3x^2 y^2 = \frac{3}{4} \\
\frac{\partial x}{\partial s} &= \cos t = \frac{1}{\sqrt{2}} \\
\frac{\partial x}{\partial t} &= -s \sin t = -\frac{1}{\sqrt{2}} \\
\frac{\partial y}{\partial s} &= \sin t = \frac{1}{\sqrt{2}} \\
\frac{\partial y}{\partial t} &= s \cos t = \frac{1}{\sqrt{2}}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = \frac{3}{4} \cdot \frac{1}{\sqrt{2}} + \frac{3}{4} \cdot \frac{1}{\sqrt{2}} = \frac{5}{4\sqrt{2}} \\
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = \frac{3}{4} \cdot \frac{1}{\sqrt{2}} + \frac{3}{4} \cdot \frac{1}{\sqrt{2}} = \frac{1}{4\sqrt{2}}
\end{align*}
\]
General case: \[ z = f(x_1, x_2, \ldots, x_n) \]

Each \( x_i \) depends on \( (t_1, \ldots, t_m) \): \[ x_i = g_i(t_1, \ldots, t_m) \]

Then \( z \) depends indirectly on \( (t_1, \ldots, t_m) \) through \( (x_1, \ldots, x_n) \)

And
\[ \frac{\partial z}{\partial t_1} = \frac{\partial z}{\partial x_1} \frac{\partial x_1}{\partial t_1} + \frac{\partial z}{\partial x_2} \frac{\partial x_2}{\partial t_1} + \ldots + \frac{\partial z}{\partial x_n} \frac{\partial x_n}{\partial t_1} \]

\[ \frac{\partial z}{\partial t_2} = \frac{\partial z}{\partial x_1} \frac{\partial x_1}{\partial t_2} + \frac{\partial z}{\partial x_2} \frac{\partial x_2}{\partial t_2} + \ldots + \frac{\partial z}{\partial x_n} \frac{\partial x_n}{\partial t_2} \]

\[ \vdots \]

\[ \frac{\partial z}{\partial t_m} = \frac{\partial z}{\partial x_1} \frac{\partial x_1}{\partial t_m} + \frac{\partial z}{\partial x_2} \frac{\partial x_2}{\partial t_m} + \ldots + \frac{\partial z}{\partial x_n} \frac{\partial x_n}{\partial t_m} \]

(This would look prettier in the language of matrices!)

Can organize the dependences in terms of a flow diagram:

E.g.: \[ y = y(x), \quad x = x(t), \quad z = z(x, y) \]

\[ \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial z}{\partial z} \frac{\partial z}{\partial s} \]
Multivariable chain rule gives another way of understanding implicit differentiation.

Recall: If we have a curve in the plane defined by \( F(x, y) = 0 \) \((\star)\)

it determines \( y \) as an implicit function of \( x \). \( y = y(x) \)

Q: what is \( \frac{dy}{dx} \)?

\[
F(x, y(x)) = 0
\]

\[
\frac{d}{dx} F(x, y(x)) = 0
\]

\[
\frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}
\]