Last time: Chain rule for multivariable functions

\[ F(x, y, z) \quad x = x(s, t) \]
\[ y = y(s, t) \]
\[ z = z(s, t) \]

\[
\frac{dF}{ds} = \frac{\partial F}{\partial x} \frac{dx}{ds} + \frac{\partial F}{\partial y} \frac{dy}{ds} + \frac{\partial F}{\partial z} \frac{dz}{ds}
\]

Implicit diff. via the chain rule:
Suppose we have \( F(x, y, z) = 0 \) determining \( z = z(x, y) \)
and want to determine \( \frac{dz}{dx}, \frac{dz}{dy} \).

\[
F(x, y, z(x, y)) = 0
\]

Now define a new function of two variables \( G(x, y) = F(x, y, z(x, y)) = 0 \)

Let's compute \( \frac{\partial}{\partial x} G(x, y) \) by the chain rule.

We get:
\[
\frac{\partial}{\partial x} G(x, y) = \frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dx} + \frac{\partial F}{\partial z} \frac{dz}{dx}
\]

\[
= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \left( \frac{1}{\frac{dz}{dx}} \right) + \frac{\partial F}{\partial z} \frac{dz}{dx}
\]

\[
= 0 \quad \text{because } G(x, y) = 0
\]

so we get:
\[
0 = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \frac{dz}{dx}
\]

i.e.
\[
\frac{dz}{dx} = - \frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}
\]
Directional Derivatives (Ch 14.6)

We've talked a lot about $f_x, f_y$

$f_x = \text{"rate of change in the } x\text{-dir"} \quad \rightarrow$

$f_y = \text{"rate of change in the } y\text{-dir"} \quad \uparrow$

How about rate of change in some other direction?

To specify a direction, pick a unit vector in that direction $\vec{u}$

- $x$-dir: $\vec{u} = \langle 1, 0 \rangle$
- $y$-dir: $\vec{u} = \langle 0, 1 \rangle$

Say $\vec{u} = \langle a, b \rangle$

Then define $D_{\vec{u}}f(x,y) = \lim_{h \to 0} \frac{f(x+a \cdot h, y+b \cdot h) - f(x,y)}{h}$

"directional derivative of $f$ in the direction $\vec{u}$"

E.g. if $\vec{u} = \langle 1, 0 \rangle$, $D_{\vec{u}}f = f_x$

if $\vec{u} = \langle 0, 1 \rangle$, $D_{\vec{u}}f = f_y$.

Fact (If $f$ is differentiable), if $\vec{u} = \langle a, b \rangle$

$$D_{\vec{u}}f = a \cdot f_x + b \cdot f_y$$

Why? Say $g(h) = f(x+ah, y+bh)$

Then $D_{\vec{u}}f(x,y) = \lim_{h \to 0} \frac{f(x+ah, y+bh) - f(x,y)}{h} = \lim_{h \to 0} \frac{g(h)-g(0)}{h} = g'(0)$

By chain rule, $\frac{d}{dh} g(h) = \frac{d}{dh} f(x+ah, y+bh) = f_x(x+ah, y+bh) \cdot a + f_y(x+ah, y+bh) \cdot b$
\[ g'(0) = f_x(x,y) \cdot a + f_y(x,y) \cdot b \]

**Example**

If \( f(x,y) = x^3 - xy + 4y^2 \)

and \( \mathbf{u} \) is unit vector with angle \( \frac{\pi}{4} \) to \( x \)-axis.

What is \( D_\mathbf{u} f(1,2) \)?

\[ \mathbf{u} = \left\langle \cos \frac{\pi}{4}, \sin \frac{\pi}{4} \right\rangle = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \]

\( f_x = 3x^2 - y \quad f_x(1,2) = 1 \)

\( f_y = -x + 8y \quad f_y(1,2) = 15 \)

\[ D_\mathbf{u} f(1,2) = a \cdot f_x(1,2) + b \cdot f_y(1,2) \]

\[ \frac{1}{\sqrt{2}} \cdot 1 + \frac{1}{\sqrt{2}} \cdot 15 = \frac{16}{\sqrt{2}} = \frac{8\sqrt{2}}{2} = 8\sqrt{2} \]

**Gradient vector**

Note \( D_\mathbf{u} f = a \cdot f_x + b \cdot f_y \)

\[ = \left\langle a, b \right\rangle \cdot \left\langle f_x, f_y \right\rangle \]

\[ = \mathbf{u} \cdot \left\langle f_x, f_y \right\rangle \]

So, define \( \nabla f = \left\langle f_x, f_y \right\rangle \) "gradient vector of \( f \)"

Then \( D_\mathbf{u} f = \mathbf{u} \cdot \nabla f \)

**Example**

If \( f(x,y) = e^x y^2 \)

then \( \nabla f = \left\langle f_x, f_y \right\rangle = \left\langle e^x y^2, 2e^x y \right\rangle \)
\((x,y)=(0,2):\) 
\(\nabla f(0,2) = \langle 4, 4 \rangle\)
\(\nabla f(0,1) = \langle 1, 2 \rangle\)
\(\nabla f(0,0) = \langle 0, 0 \rangle\)

**Ex.** What is the directional derivative of \(f(x,y) = e^x y^2\) at \((0,2)\)
in the direction of the vector \(\langle -3, 4 \rangle\)?

\[
\vec{u} = \frac{\langle -3, 4 \rangle}{\|\langle -3, 4 \rangle\|} = \frac{\langle -3, 4 \rangle}{5} = \langle -\frac{3}{5}, \frac{4}{5} \rangle
\]

\[
D_{\vec{u}} f = \langle -\frac{3}{5}, \frac{4}{5} \rangle \cdot \langle 4, 4 \rangle = -\frac{12}{5} + \frac{16}{5} = \frac{4}{5}
\]

We did all this in 2 dimensions \(f(x,y)\)
but all works just the same in 3d \(f(x,y,z)\)

\[
e.g. \quad \vec{u} = \langle a, b, c \rangle \quad D_{\vec{u}} f(x,y,z) = \lim_{h \to 0} \frac{f(x+ah, y+bh, z+ch) - f(x,y,z)}{h}
\]

\[
= \vec{u} \cdot \nabla f
\]

\[\nabla f = \langle f_x, f_y, f_z \rangle\]

**Ex.** If \(f(x,y,z) = \frac{y^2}{x}\) find \(\nabla f\) at \((-2,3,5) = (x,y,z)\)
and the dir. deriv. of \(f\) in direction \(\langle 1, 2, 0 \rangle\)

\[
\nabla f = \langle -\frac{y^2}{x^2}, 2yz, \frac{y^2}{x^2} \rangle \quad \text{s.} \quad \nabla f (-2,3,5) = \langle -\frac{45}{4}, -15, -\frac{9}{2} \rangle
\]

\[
\vec{u} = \frac{\langle 1, 2, 0 \rangle}{\|\langle 1, 2, 0 \rangle\|} = \frac{\langle 1, 2, 0 \rangle}{\sqrt{5}}
\]
At \((-2,3,5)\)
\[
\nabla f = \vec{u} \cdot \nabla f = \left< \frac{120}{\sqrt{5}}, \frac{-45}{\sqrt{5}}, \frac{-9}{\sqrt{5}} \right> = \frac{1}{\sqrt{5}} \left( -\frac{45}{4} - 30 + 0 \right) = \frac{1}{\sqrt{5}} \left( -\frac{165}{4} \right) = -\frac{165}{4\sqrt{5}}
\]

**How to interpret/think about \(\nabla f\)?**

- **Fact**
  - \(\nabla f\) is always \(\perp\) to contour lines.
  - *Why?* If we walk along a path \(\vec{t}\) tangent to a contour line, then \(f\) is constant. So, if \(\vec{t}\) is tangent to a contour line, then \(D_{\vec{t}} f = 0\). But \(D_{\vec{u}} f = \vec{u} \cdot \nabla f = 0\) so \(\vec{u} \perp \nabla f\).

- **Fact** The unit vector \(\vec{u}\) for which \(D_{\vec{u}} f\) is maximum is the vector \(\vec{u} = \frac{\nabla f}{\|\nabla f\|}\) (if \(\nabla f \neq \left< 0,0 \right>\))

("\(\nabla f\) tells you which way to walk to increase \(f\) the fastest")
Why? \[ D_u f = \vec{u} \cdot \nabla f \]
\[ = \|\vec{u}\| \cdot \|\nabla f\| \cdot \cos \theta \]
\[ = \|\nabla f\| \cdot \cos \theta \]
maximized when \( \cos \theta = 1 \), i.e. \( \theta = 0 \)
i.e. \( \vec{u} \) is parallel to \( \nabla f \).
i.e. \( \vec{u} = \frac{\nabla f}{\|\nabla f\|} \)

Similarly \( -\frac{\nabla f}{\|\nabla f\|} = \vec{u} \) gives the direction of fastest decrease.

**Fact** If \( f = f(x,y,z) \) have level surfaces in 3D (rather than level curves in 2D) and again, \( \nabla f \) is \( \perp \) to the level surfaces.

**Ex** Find the tangent plane to the ellipsoid
\[ \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{9} = 3 \]
at \((x,y,z) = (-2,1,-3)\).

How to find a normal vector to this plane?

The ellipsoid is a level surface: \( F(x,y,z) = \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{9} = 3 \)
Thus $\nabla F$ is $\perp$ to the ellipsoid $\Rightarrow$ also $\perp$ to tangent plane $\nabla F = \langle \frac{x}{2}, 2y, \frac{2}{3}z \rangle$ 
$\nabla F(-2, 1, -3) = \langle -1, 2, -\frac{2}{3} \rangle$ 
So, we want plane through $(-2, 1, -3)$ w/ normal vector $\langle -1, 2, -\frac{2}{3} \rangle$

$-1(x+2) + 2(y-1) - \frac{2}{3}(z+3) = 0$
$-x - 2 + 2y - 2 - \frac{2}{3}z - 2 = 0$
$-x + 2y - \frac{2}{3}z = 6$

Rk
$f(x, y) = x^2 - y^2$ 
$\nabla f = \langle 2x, -2y \rangle$

Illustrates th' phenomena: if contour lines cross, $\nabla f \neq 0$ there.