Maxima and Minima for functions of 2 variables (Ch 14.7)

**Def:** If we have a function \( f(x,y) \)
then we say \((a,b)\) is a *local minimum* (or *maximum*) for \( f \) if there is some \( \varepsilon \) such that, for all points \((x,y)\) in a disk of radius \( \varepsilon \) around \((a,b)\),
\[
\begin{align*}
  f(x,y) &\leq f(a,b) \quad \text{(or } f(x,y) \geq f(a,b) \text{ for max)}
\end{align*}
\]

Recall the picture in 1 variable:

- Local minimum
- Local maximum

Cartoon map:

1. Local minimum
2. Local maximum
3. 4. Local minimum
   5. Local maximum
Def: If \( \frac{\partial^2 f}{\partial x^2} = 0 \) and \( \frac{\partial^2 f}{\partial y^2} = 0 \)
then we call \((a,b)\) a critical point of \(f\).

Fact: (If \( f \) is differentiable), if \((a,b)\) is a local min/max for \( f \),
then \((a,b)\) is a critical point for \( f \).

Ex: Find the local minima and maxima of
\[ f(x,y) = x^2 + y^2 - 2x - 6y + 14. \]

Find critical points:
\[
\begin{align*}
\frac{\partial f}{\partial x} &= 2x - 2 = 0 \quad f_x = 2x - 2 = 0 \\
\frac{\partial f}{\partial y} &= 2y - 6 = 0 \quad f_y = 2y - 6 = 0
\end{align*}
\]
Solve for \((x,y)\):
\[ x = 1, \ y = 3 \]
\[
\Rightarrow \text{ only one critical point, at } (1,3). \\
\text{Local min - e.g. from visualizing the graph} \\
\text{or rewrite} \quad f(x,y) = (x-1)^2 + (y-3)^2 + 4
\]

Ex: Find local min/local max for
\[ f(x,y) = x^2 - y^2 \]

Critical pts:
\[ \begin{align*}
\frac{\partial f}{\partial x} &= 2x = 0 \\
\frac{\partial f}{\partial y} &= -2y = 0
\end{align*} \]

Solve for \((x,y)\): \((x,y) = (0,0)\)
(0,0) is a saddle point (cf. last lecture)
not a local max/min.

So if \( f(xy) \) has no local max/min.

Second derivative test \((\text{If } f_{xx}, f_{yy}, f_{xy} \text{ are continuous near } (a,b))\)

If \((a,b)\) is a critical point of \(f:\)

let \( D = f_{xx}f_{yy} - f_{xy}^2 \)

\[ \begin{align*}
\text{If } D > 0, & \quad \begin{cases} 
\text{if } f_{xx} > 0 \text{ then } (a,b) \text{ is local min} \\
\text{if } f_{xx} < 0 \text{ then } (a,b) \text{ is local max}
\end{cases} \\
\text{If } D < 0, & \quad (a,b) \text{ is saddle point} \\
\text{If } D = 0, & \quad \text{test is inconclusive}
\end{align*} \]

\[ D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} \]

\[ \begin{align*}
\text{Ex} \quad & \text{If } f(xy) = x^2 + y^2 - 2x - 6y + 14 \\
& f_x = 2x - 2 \quad f_y = 2y - 6 \\
& f_{xx} = 2 \quad f_{xy} = 0 \quad f_{yy} = 2 \\
& (x,y) = (1,3) \quad \text{cntr. pt.} \\
\end{align*} \]

\[ D = f_{xx}f_{yy} - f_{xy}^2 = 2 \cdot 2 - 0 = 4 > 0 \]

and \( f_{xx} = 2 > 0 \)

\( \Rightarrow \) \((1,3)\) is local minimum

\[ \text{Ex} \quad & \text{If } f(xy) = x^2 + 3y^2 - 10xy \\
& f_x = 2x - 10y \quad f_y = 6y - 10x \]
\[ f_{xx} = 2 \quad f_{xy} = -10 \quad f_{yy} = 6 \]

critical point: \[ 2x - 10y = 0 \]
\[ 6y - 10x = 0 \]

solve for \( (x, y) \): \( x = 0, \ y = 0 \)

\( \Rightarrow \) only critical point is \((0, 0)\).

2nd derv test: \[ D = \begin{vmatrix} 2 & -10 \\ -10 & 6 \end{vmatrix} = 12 - 100 = -88 < 0 \]

\( \Rightarrow (0, 0) \) is saddle point.

Why does 2nd derv test work?

\[ e.g., \text{suppose } D > 0, \ f_{xx} > 0. \]
\[ \bar{u} = \langle h, k \rangle \]

\[ D_{\bar{u}} f = h \cdot f_x + k \cdot f_y = 0 \text{ at critical pt.} \]

\[ \bar{D}_{\bar{u}} \bar{D}_{\bar{u}} f = h \cdot (h f_{xx} + k f_{yx}) + k \cdot (h f_{xy} + k f_{yy}) \]
\[ = h^2 f_{xx} + h k f_{yx} + h k f_{xy} + k^2 f_{yy} \]
\[ = h^2 f_{xx} + 2 h k f_{xy} + k^2 f_{yy} \]
\[ = f_{xx} \left( h + \frac{f_{xy}}{f_{xx}} k \right)^2 + \frac{k^2}{f_{xx}} \left( f_{xx} f_{yy} - f_{xy}^2 \right) \]
\[ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \]
\[ > 0 \quad \geq 0 \quad \geq 0 \quad \geq 0 \quad \geq 0 \quad \geq 0 \quad > 0 \quad > 0 \]

\( \Rightarrow (0, 0) \) is a local minimum!
Often we want global max/min, not local.

**Def.** If there is a function $f(x,y)$ defined on some domain $D$ in $(x,y)$-plane, we say $(a,b)$ is a **global maximum** for $f$ if, for any $(x,y)$ in $D$, $f(a,b) \geq f(x,y)$.

**Fact.** If $D$ is **bounded** (does not go off to $\infty$ in any direction) and **closed** (contains all its boundary points), and $f$ is continuous on $D$, then $f$ has a global max, and a global min, on $D$.

**Ex.** $D = \{(x,y): x^2 + y^2 \leq 1\}$ is bounded and closed.

$D = \{(x,y): |x| \leq 1\}$ is closed, not bounded.

$D = \{(x,y): x^2 + y^2 < 1\}$
So, suppose $D$ is closed and bounded.

To find the global maximum of $f$ on $D$:

1. Find all critical points of $f$ on the interior of $D$, find the values of $f$ at these points.

2. Find the maximum value of $f$ on the boundary of $D$.

3. Take the biggest value of $f$ found in steps 1, 2.

(Similarly for absolute minimum.)

**Ex** $f(x, y) = x^2 - 2xy + 2y$

Find absolute min, max of $f(x, y)$ on $D = \{0 \leq x \leq 3, 0 \leq y \leq 2\}$

1. **Critical pts:** $f_x = 2x - 2y = 0$ $f_y = -2x + 2 = 0$ 
   $\rightarrow$ $x = 1, y = 1$

   One crit pt, at $(1,1)$ $f(1,1) = 1$
(2) On boundary: 4 pieces

$L_1: y = 0, \ 0 \leq x \leq 3$

\[ f(x, 0) = x^2 \]

\[ \min: \ f(0, 0) = 0 \]

\[ \max: \ f(3, 0) = 9 \]

$L_2: x = 3, \ 0 \leq y \leq 2$

\[ f(3, y) = 9 - 4y \]

\[ \min: \ f(3, 2) = 1 \]

\[ \max: \ f(3, 0) = 9 \]

$L_3: y = 2, \ 0 \leq x \leq 3$

\[ f(x, 2) = x^2 - 4x + 4 \]

Need to find min, max of this for \(0 \leq x \leq 3\)

Now just a function of one variable \(x\)

\[ f' = 2x - 4 \rightarrow \text{crit pt at } x = 2 \]

\[ f(2, 2) = 0 \]

Also

\[ f(0, 2) = 4 \]

\[ f(3, 2) = 1 \]

$L_4: x = 0, \ 0 \leq y \leq 2$

\[ f(0, y) = 2y \]

\[ f(0, 0) = 0 \]

\[ f(0, 2) = 4 \]

\[ \Rightarrow \text{global max is } 9: \ f(3, 0) = 9 \]

\[ \text{global min is } 0: \ f(0, 0) = 0 \]

\[ f(2, 2) = 0 \]

An application: least-squares fitting.

Say we have some data points and want to find the "best-fit line."
Consider squared error \[ E = (y_1 - mx_1 - b)^2 + (y_2 - mx_2 - b)^2 + \ldots + (y_n - mx_n - b)^2 \]

"Best-fit line": the line which minimizes \( E \)

What is the best-fit line to these 3 data points?

\[ E = (2-b)^2 + (4-m-1)^2 + (3-2m-b)^2 \]

\[ = (4-4b+b^2)+(16+m^2+1^2-8m-8b+2bm)+(9+4m^2+b^2-12m-6b+4bm) \]

\[ = 3b^2 + 5m^2 + 6bm - 20m - 18b + 29 \]

Critical points:

\[ E_b = 6b+6m-18 = 0 \]

\[ E_x = 6b+10m-20 = 0 \]

\[ -4m + 2 = 0 \quad \rightarrow \quad m = \frac{1}{2} \]

\[ 6b + 6\left(\frac{1}{2}\right) - 18 = 0 \]

\[ 6b - 15 = 0 \quad \rightarrow \quad b = \frac{5}{2} \]

So best-fit line is \( y = \frac{1}{2}x + \frac{5}{2} \)