Theorem 1. *Any polygonal disk with edges identified in pairs is homeomorphic to a compact, connected, triangulated* 2*-manifold.*

Theorem 2. *Any compact, connected, triangulated* 2*-manifold is homeomorphic to a polygonal disk with edges identified in pairs.*

Exercise 3. *The boundary of a tetrahedron is naturally triangulated with a triangulation T consisting of four* 2*-simplexes together with their six edges and four vertices. On the boundary of a tetrahedron locate the first and second derived subdivisions of T, the* 1*-skeleton of T, the regular neighborhood of the* 1*-skeleton of T, the regular neighborhoods of a vertex and an edge of T, and the dual* 1*-skeleton of T.*

Exercise 4. *On the accompanying pictures of the second derived subdivisions of triangulations of the torus and the Klein bottle, find regular neighborhoods of subsets of the* 1*-skeleton.*

Exercise 5. *Characterize graphs in the* 1*-skeleton of T for the triangulations of the sphere, torus, and projective plane whose regular neighborhoods are homeomorphic to a disk.*

Theorem 6. *Let M^2 be a compact, triangulated* 2*-manifold with triangulation T. Let S be a tree whose edges are* 1*-simplices in the* 1*-skeleton of T. Then $N(S)$, the regular neighborhood of S, is homeomorphic to $\mathbb{D}^2$.*

Theorem 7. *Let M^2 be a compact, triangulated* 2*-manifold with triangulation T. Let S be a tree equal to a union of edges in the dual* 1*-skeleton of T. Then $\cup\{\sigma''_j \mid \sigma''_j \in T''$ and $\sigma''_j \cap S \neq \emptyset\}$ is homeomorphic to $\mathbb{D}^2$.*

Theorem 8. *Let M^2 be a connected, compact, triangulated* 2*-manifold with triangulation T. Let S be a tree in the* 1*-skeleton of T. Let S' be the subgraph of the dual* 1*-skeleton of T whose edges do not intersect S. Then S' is connected.*

The following two theorems state that M^2 can be divided into two pieces, one a disk D_0, and the other a disk (D_1) with bands (the H_i's) attached to it.

Theorem 9. *Let M^2 be a connected, compact, triangulated* 2*-manifold. Then $M^2 = D_0 \cup D_1 \cup \left(\bigcup_{i=1}^{k} H_i\right)$ where D_0, D_1, and each H_i is homeomorphic to $\mathbb{D}^2$, $\text{Int } D_0 \cap D_1 = \emptyset$, the H_i's are disjoint, $\bigcup_{i=1}^{k} \text{Int } H_i \cap (D_0 \cup D_1) = \emptyset$, and for each i, $H_i \cap D_1$ equals* 2 *disjoint arcs each arc on the boundary of each of H_i and D_1.*

Theorem 10. *Let M^2 be a connected, compact, triangulated 2-manifold. Then:*

1. *There is a disk D_0 in M^2 such that $M^2 - (\text{Int}\, D_0)$ is homeomorphic to the following subset of $\mathbb{R}^3$: a disk D_1 with a finite number of disjoint strips, H_i for $i \in \{1, \dots n\}$, attached to boundary of D_1 where each strip has no twist or $1/2$ twist. (See example below.)*

2. *Furthermore, the boundary of the disk with strips, $D_1 \cup \left(\bigcup_{i=1}^{k} H_i\right)$, is connected.*

untwisted pairs *twisted strips*

Exercise 11. *In the set-up in the previous theorem, any strip H_i divides the boundary of D_0 into two edges e_i^1 and e_i^2, where H_i is* not *attached. Show that if a strip H_j is attached to D_0 with no twists, then there must be a strip H_k that is attached to both e_j^1* and *e_j^2.*

Theorem 12. *Let M^2 be a connected, compact, triangulated 2-manifold. Then there is a disk D_0 in M^2 such that $M^2 - \text{Int}\ D_0$ is homeomorphic to a disk D_1 with strips attached as follows: first come a finite number of strips with $1/2$ twist each whose attaching arcs are consecutive along $\text{Bd}\, D_1$, next come a finite number of pairs of untwisted strips, each pair with attaching arcs entwined as pictured with the four arcs from each pair consecutive along $\text{Bd}\, D_1$.*

Theorem 13. *Let X be a disk D_0 with one strip attached with a $1/2$ twist with its attaching arcs consecutive along $\text{Bd}\, D_0$ and one pair of untwisted strips with attaching arcs entwined as pictured with the four arcs consecutive* *along* $\mathrm{Bd}\, D_0$. *Let* Y *be a disk* D_1 *with three strips with a* $1/2$ *twist each whose attaching arcs are consecutive along* $\mathrm{Bd}\, D_1$. *Then* X *is homeomorphic to* Y.

X Y

Theorem 14. *Let* M^2 *be a connected, compact, triangulated* 2*-manifold. Then there is a disk* D_0 *in* M^2 *such that* $M^2 - \mathrm{Int}\, D_0$ *is homeomorphic to one of the following:*

a) a disk D_1,

b) a disk D_1 *with* k $\frac{1}{2}$*-twisted strips with consecutive attaching arcs, or*

c) a disk D_1 *with* k *pairs of untwisted strips, each pair in entwining position with the four attaching arcs from each pair consecutive.*

entwining pair of strips

Theorem 15 (Classification of compact, connected 2-manifolds)**.** *Any connected, compact, triangulated* 2*-manifold is homeomorphic to the* 2*-sphere* $\mathbb{S}^2$, *a connected sum of tori, or a connected sum of projective planes.*