Name:
Date:
Group number mod 6:

Homework 3 (Chapter 2 2.25-2.49)
Everyone must do the starred problems and you must do your own problems mod 6.

0.1 Invariants

0.1.1 Euler characteristic

Theorem 1. ** Let M^2 be a connected, compact, triangulated 2-manifold with triangulation T. Let T' be a subdivision of T. Then $\chi(M^2, T) = \chi(M^2, T')$.*

Theorem 2. ** Let M_1^2 and M_2^2 be connected, compact, triangulated 2-manifolds. If M_1^2 is PL-homeomorphic to M_2^2, then $\chi(M_1^2) = \chi(M_2^2)$.*

Theorem 3.

1. $\chi(\mathbb{S}^2) = 2$.
2. $\chi(\mathbb{T}^2) = 0$.
3. $\chi(\mathbb{R}\mathrm{P}^2) = 1$.
4. $\chi(\mathbb{K}^2) = 0$.

Theorem 4. ** Let M_1^2 and M_2^2 be two connected, compact, triangulated 2-manifolds. Then $\chi(M_1^2 \# M_2^2) = \chi(M_1^2) + \chi(M_2^2) - 2$.*

Theorem 5. *Let $\mathbb{T}_i^2$ be the torus for $i = 1, \ldots, n$. Then*

$$\chi\left(\mathop{\#}_{i=1}^{n} \mathbb{T}_i^2\right) = 2 - 2n.$$

Theorem 6. *Let $\mathbb{R}\mathrm{P}_i^2$ be the projective plane for $i = 1, \ldots, n$. Then*

$$\chi\left(\mathop{\#}_{i=1}^{n} \mathbb{R}\mathrm{P}^2\right) = 2 - n.$$

0.1.2 Orientability

Exercise 7. *Show that the induced orientation on an edge of a 2-simplex is well defined; in other words, that it is independent of the choice of positive equivalence class representative.*

Theorem 8. *Suppose (M^2, T) is a 2-manifold with triangulation T and T' is a subdivision of T. Then if (M^2, T) is orientable, so is (M^2, T').*

Theorem 9. *Orientability is preserved under PL homeomorphism.*

Theorem 10. *M^2 is orientable if and only if it contains no Möbius band.*

Theorem 11. ** Let $M = M_1 \# \ldots \# M_n$. Then M is orientable if and only if M_i is orientable for each $i \in \{1, \ldots, n\}$.*

Theorem 12 (Classification of compact, connected 2-manifolds)**.** ** If M^2 is a connected, compact, triangulated 2-manifold then:*

(a) if $\chi(M^2) = 2$, then $M^2 \cong \mathbb{S}^2$.

(b) if M^2 is orientable and $\chi(M^2) = 2 - 2n$, for $n \geq 1$, then

$$M^2 \cong \left(\mathop{\#}_{i=1}^{n} T_i^2 \right) .$$

(c) if M^2 is non-orientable and $\chi(M^2) = 2 - n$, for $n \geq 1$, then

$$M^2 \cong \left(\mathop{\#}_{i=1}^{n} \mathbb{R}\mathrm{P}_i^2 \right) .$$

Problem 13. ** Identify the following 2-manifolds as a sphere, or a connected sum of n tori (specifying n), or a connected sum of n projective planes (specifying n).*

a. $\mathbb{T}\#\mathbb{R}\mathrm{P}$

b. $\mathbb{K}\#\mathbb{R}\mathrm{P}$

c. $\mathbb{R}\mathrm{P}\#\mathbb{T}\#\mathbb{K}\#\mathbb{R}\mathrm{P}$

d. $\mathbb{K}\#\mathbb{T}\#\mathbb{T}\#\mathbb{R}\mathrm{P}\#\mathbb{K}\#\mathbb{T}$

0.2 CW complexes

Theorem 14. *Let* (M^2,T) *be a triangulated* 2*-manifold. Suppose* $\sigma = \{uvw\}$ *and* $\sigma' = \{uvw'\}$ *are two distinct* 2*-simplexes in* T *that share the edge* $e = \{uv\}$*. Then we can create a new structure for* M^2 *alternative to* T*, namely,* P *where* $P = T \cup \{\tau\} - \{\sigma, \sigma', e\}$*, where* $\tau = \sigma \cup \sigma'$ *is the polygon formed by the union of the two* 2*-simplices along their shared edge. If* v'*,* e'*,* f' *are the numbers of vertices, edges, and polygons in* P*, then the Euler Characteristic* $\chi(M^2,T) = v' - e' + f'$*.*

Theorem 15. *Let* (M^2,T) *be compact, triangulated* 2*-manifold with Euler characteristic* $\chi(M^2,T)$*. Suppose we create a polygonal structure* P *on* M^2 *inductively as follows. Let* $P_0 = T$*. Suppose we have created* P_i*. Suppose two* 2*-dimensional objects* σ *and* σ' *in* P_i *share a connected path of edges in the boundary of each from vertex* u *to* w *(*$v \neq w$*). We create* P_{i+1} *by removing* σ *and* σ' *from* P_i*, removing all the edges in the path from vertex* u *to* w*, removing all vertices of the edges in that path except for* u *and* w*, and putting in the single two dimensional object* $\sigma \cup \sigma'$*. Then if* v*,* e*,* f *are the numbers of vertices, edges, and* 2*-dimensional objects in* P_{i+1}*, then* $\chi(M^2,T) = v - e + f$*.*

Theorem 16. ** Let* (M^2,T) *be compact, triangulated* 2*-manifold with a polygonal structure* P *as defined inductively in the previous theorem. Suppose we substitute* P *with a new structure obtained inductively as follows.*

Let $P = P_0$. If P_i has an edge e with a free vertex v, that is, v is not the boundary of any other edge in P_i, then remove v and e from P_i to create P_{i+1}. If P_i has a vertex v that is one end of an edge e in P_i and one end of an edge f in P_i and v is not on the end of any other edge, then remove v, e, and f from P_i and put in the new 1-dimensional object $e \cup f$ to create P_{i+1}. Then if v', e', f' are the numbers of vertices, 1-dimensional objects, and 2-dimensional objects in an inductively defined P, then $\chi(M^2, T) = v' - e' + f'$.

Exercise 17. *Start with a triangulation of $\mathbb{S}^2$ and carry out the preceding process as far as possible. What "structure" do you get? Confirm that you get the right Euler Characteristic.*

Exercise 18. ** Start with a triangulation of $\mathbb{T}^2$ and carry out the preceding process as far as possible. What "structure" do you get? Confirm that you get the right Euler characteristic.*

Theorem 19. *Let (M^2, T) be a compact, triangulated 2-manifold with triangulation T. Then M^2 equals the disjoint union of the $\operatorname{Int} \sigma_i$ where $\sigma_i \in T$.*

Theorem 20. **Let S be a cellular decomposition of a compact, triangulated 2-manifold (M^2, T). If v, e, and f are the number of 0, 1 and 2 cells in S, then the Euler Characteristic $\chi(M^2, T) = v - e + f$.*

Problem 21. *Identify the following surfaces:*

a. *The surface obtained by identifying the edges of the octagon as indicated:*

b. *The surface obtained by identifying the edges of the decagon as indicated:*

0.3 2-manifolds with boundary

Exercise 22. *What should be the definition of a connected, compact, triangulated* 2*-manifold-with-boundary?*

Exercise 23. *Formulate the necessary definitions and theorem statements that classify compact, connected, triangulated* 2*-manifolds-with-boundary. Prove your theorems.*

Problem 24. *Identify the following surfaces made by two disks joined by bands as indicated:*

a. n *twisted bands*

b. 1 *untwisted band and* $n-1$ *twisted bands*

Exercise 25. ** Fill out the following table, using the connected sum decom-*

position. The number of boundary components is denoted by $|\partial|$.

$|\partial|$	0		1		2		3	
χ	*orient.*	*non-or.*	*orient.*	*non-or.*	*orient.*	*non-or.*	*orient.*	*non-or.*
2								
1								
0								
-1								
-2								
-3								
-4								
-5								