Chapter 3

Fundamental group and covering spaces

3.1 Fundamental group

Theorem 1. ** Given topological spaces X and Y, $\simeq$ is an equivalence relation on the set of all continuous functions from X to Y.*

Theorem 2. *Given topological spaces X and Y with $S \subset X$, being homotopic relative to S is an equivalence relation on the set of all continuous functions from X to Y.*

Theorem 3. ** Let α be a loop into the topological space X. Then $\alpha = \beta \circ \omega$ where ω is the standard wrapping map and β is a continuous function from $\mathbb{S}^1$ into X.*

Theorem 4. *Let X be a topological space and let p be a point in X. Then a loop $\alpha = \beta \circ \omega$ (where ω is the standard wrapping map and β is a continuous function from $\mathbb{S}^1$ into X) is homotopically trivial if and only if β can be extended to a continuous function from $\mathbb{B}^2$ into X.*

Theorem 5. *If $\alpha \sim \alpha'$ and $\beta \sim \beta'$, then $\beta \cdot \alpha \sim \beta' \cdot \alpha'$.*

Theorem 6. *Given α, β, and γ, then $(\alpha\cdot\beta)\cdot\gamma \sim \alpha\cdot(\beta\cdot\gamma)$ and $([\alpha]\cdot[\beta])\cdot[\gamma] \sim [\alpha]\cdot([\beta]\cdot[\gamma])$.*

Theorem 7. *Let α be a path with $\alpha(0) = x_0$, then $\alpha \cdot \alpha^{-1} \sim e_{x_0}$, where e_{x_0} is the constant path $e_{x_0} : [0,1] \to x_0$. Stated differently, if α is a path, then $\alpha \cdot \alpha^{-1}$ is homotopically trivial.*

Theorem 8. ** The fundamental group $\pi_1(X, x_0)$ is a group. The identity element is the class of homotopically trivial loops based at x_0.*

Theorem 9. ** If X is path connected, then $\pi_1(X, p) \cong \pi_1(X, q)$ for any points $p, q \in X$.*

Exercise 10. *We will use 1 to denote the trivial group:*

1. $\pi_1([0, 1]) \cong 1$.
2. $\pi_1(\mathbb{S}^0, 1) \cong 1$ *where* $\mathbb{S}^0$ *is the zero-dimensional sphere* $\{-1, 1\} \subset \mathbb{R}^1$.
3. $\pi_1($*convex set*$) \cong 1$.
4. $\pi_1($*cone*$) \cong 1$.
5. $\pi_1($*cone over Hawaiian earring*$) \cong 1$.
6. $\pi_1(\mathbb{R}^n) \cong 1$ *for* n *for* $n \geq 1$.
7. $\pi_1(\mathbb{S}^2) \cong 1$.

This is question can be substituted for two other questions.

Theorem 11. *** The fundamental group of the circle $\mathbb{S}^1$ is infinite cyclic, that is, $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$.*

If you can't get the proof to work out, remember the result. It is the first non-trivial fundament group we have seen.

3.1.1 Cartesian products

Theorem 12. ** Let (X, x_0), (Y, y_0) be path connected spaces. Then $\pi_1(X \times Y,\ (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.*

Exercise 13. *Find:*

1. $\pi_1(\mathbb{T}^2 \cong \mathbb{S}^1 \times \mathbb{S}^1)$;
2. $\pi_1(\mathbb{D}^2 \times \mathbb{S}^1)$;
3. $\pi_1(\mathbb{S}^2 \times \mathbb{S}^1)$;
4. $\pi_1(\mathbb{S}^2 \times \mathbb{S}^2)$;
5. $\pi_1(\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2)$;
6. $\pi_1(\mathbb{S}^{p_1} \times \ldots \times \mathbb{S}^{p_k})$ *where* $p_j \geq 2$ *for* $1 \leq j \leq k$.

3.1.2 Induced homomorphisms

Exercise 14. *Check that for a continuous function $f : X \to Y$, the induced homomorphism on the fundamental group f_* is well-defined.*

Theorem 15. ** If $g : (X, x_0) \to (Y, y_0)$, $f : (Y, y_0) \to (Z, z_0)$ are continuous functions, then $(f \circ g)_* = f_* \circ g_*$.*

Theorem 16. *If f, $g : (X, x_0) \to (Y, y_0)$ are continuous functions and f is homotopic to g relative to x_0, then $f_* := g_*$.*

Theorem 17. ** If $h : X \to Y$ is a homeomorphism then $h_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is a group isomorphism. So homeomorphic spaces have naturally isomorphic fundamental groups.*

3.2 Retractions and fixed points

Theorem 18. *Let A be a retract of X and let $i : A \hookrightarrow X$ be the inclusion map. Then $i_* : \pi_1(A) \to \pi_1(X)$ is injective.*

Question 19. *Give an example to show why the conclusion of the previous theorem does not follow merely from the assumption that A is a subset of X.*

Theorem 20. *Let A be a retract of X, $a_0 \in A$ and $r : A \hookrightarrow X$ the retraction. Then $r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$ is surjective.*

Theorem 21 (No Retraction Theorem for $\mathbb{D}^2$)**.** ** There is no retraction from $\mathbb{D}^2$ to its boundary.*

Theorem 22. *The identity map $i : \mathbb{S}^1 \to \mathbb{S}^1$ is not null homotopic.*

Theorem 23. *The inclusion map $j : \mathbb{S}^1 \to \mathbb{R}^2 - \mathbf{0}$ is not null homotopic.*

Theorem 24 (Brouwer Fixed-Point Theorem for $\mathbb{D}^2$)**.** ** Let $f : \mathbb{D}^2 \to \mathbb{D}^2$ be a continuous map, then there is some $x \in \mathbb{D}^2$ for which $f(x) = x$.*

This is one of the coolest theorems of the course. In addition, to proving it explain why if you droped a map of the United States on the ground (in Austin), there would be one point on the map that was exactly on top of where it actually is.

Exercise 25. *Show that $\mathbb{R}^2 - \{2 \text{ points}\}$ strong deformation retracts onto the wedge of two circles. In addition, show that $\mathbb{R}^2 - \{2 \text{ points}\}$ strong deformation retracts onto a theta curve. Are the wedge of two circles and the theta curve homeomorphic? (See Figure 3.1)*

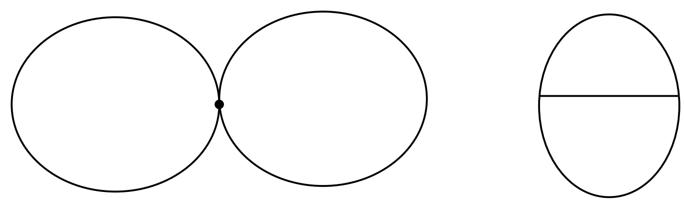

Figure 3.1: The wedge of two circles and the theta curve

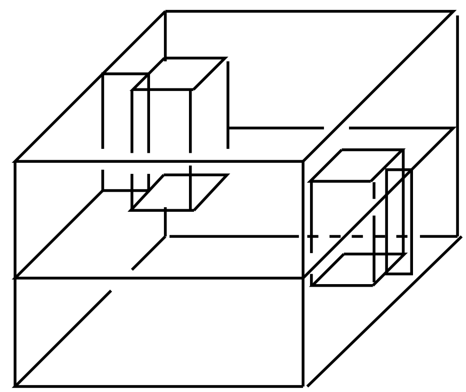

Figure 3.2: House with 2 rooms

Theorem 26. ** If $r : X \to A$ is a strong deformation retraction and $a \in A$, then $\pi_1(X, a) \cong \pi_1(A, a)$.*

Exercise 27. *Compute the fundamental groups of the following spaces:*

1. π_1(*solid torus* $\cong \mathbb{D}^2 \times \mathbb{S}^1$) $\cong$
2. $\pi_1(\mathbb{R}^2 - \mathbf{0}) \cong$
3. π_1(*house with 2 rooms*) $\cong$
4. π_1(*dunce's hat*) $\cong$

Theorem 28. *A contractible space is simply connected.*

Theorem 29. *A retract of a contractible space is contractible.*

Theorem 30. *The House with Two Rooms is contractible.*

Theorem 31. *The Dunce's Hat is contractible.*

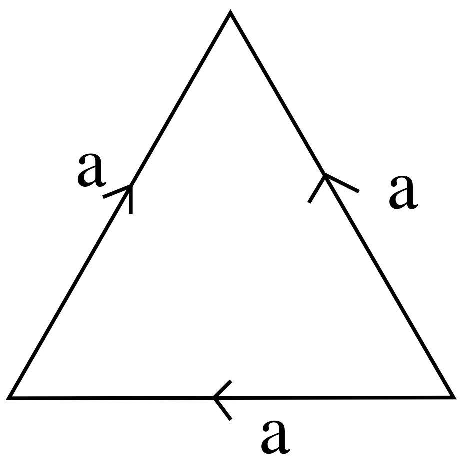

Figure 3.3: Dunce's hat