Name:
Date:
Group number mod 6:
Due: 10/5/07

Homework 5 (Chapter 3 3.32-3.51)

Everyone must do the starred problems and you must do your own problems mod 6.

3.3 Van Kampen's Theorem, I

Theorem 32. *Let $X = U \cup V$, where U and V are open and $U \cap V$ is path connected, and let $p \in U \cap V$. Then any element of $\pi_1(X, p)$ has a representative $\alpha_1\beta_1 \cdots \alpha_n\beta_n$, where each α_i is a loop in U based at p and each β_i is a loop in V based at p.*

3.3.1 Van Kampen's Theorem: simply connected intersection case

***Theorem 33** (Van Kampen's Theorem, simply connected intersection case). *Let $X = U \cup V$, where U, V are open, path connected subsets of X, $U \cap V$ is path connected and simply connected, and $x \in U \cap V$. Then $\pi_1(X, x) \cong \pi_1(U, x) * \pi_1(V, x)$.*

Corollary 34. ** Let ∞ denote the wedge of two circles. Then $\pi_1(\infty) \cong \mathbb{Z} * \mathbb{Z}$.*

Question 35. *Let X be the wedge of n circles. What is $\pi_1(X)$?*

Theorem 36. *If A and B are each connected, then $A \vee B$ is connected. If A and B are each path connected, then $A \vee B$ is path connected.*

Theorem 37. *Let X be the wedge of two cones over two Hawaiian earrings, where they are identified at the points of tangency of the circles of each Hawaiian earring, as in the figure below. Then $\pi_1(X) \ncong 1$.*

Question 38. *State conditions that suffice to ensure that $\pi_1(A \vee B) \cong \pi_1(A) * \pi_1(B)$.*

Theorem 39. *Show that π_1(Hawaiian earring) is not finitely generated, in fact, π_1(Hawaiian earring) is not countably generated.]*

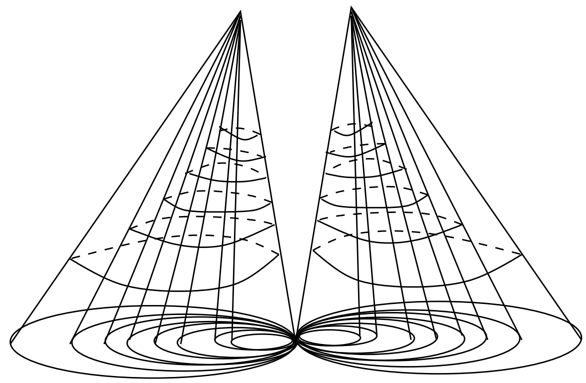

Figure 3.1: Wedge of cones over Hawaiian earrings

3.3.2 Van Kampen's Theorem: simply connected pieces case

Theorem 40 (Van Kampen's Theorem, simply connected pieces case)**.** *Let $X = U \cup V$ where U and V are open, path connected, and simply connected subsets of X and $U \cap V$ is path connected. Then X is simply connected.*

Exercise 41.

1. $\pi_1(\mathbb{S}^2) = 1$
2. $\pi_1(\mathbb{S}^n) = 1$

Question 42. *Can you find an example where U and V are simply connected, but $X = U \cup V$ is not simply connected?*

3.4 Fundamental groups of surfaces

Exercise 43. *Describe a strong deformation retract, together with its fundamental group, of a once-punctured compact, connected, triangulated 2-manifold.*

Exercise 44. *Let M^2 be a compact, connected, triangulated 2-manifold, and assume we write M^2 as $U \cup D^2$, where U and D^2 are open subspaces of M^2, D^2 is an open disk, $U \cap D^2 \simeq A^2$ is an open annulus, and $p \in A^2$. Describe the non-trivial elements of $\pi_1(U, p)$ that are trivial in $\pi_1(M^2, p)$.*

Exercise 45. *State and prove a theorem that allows you to calculate $\pi_1(M^2)$ for any compact, connected, triangulated 2-manifold M^2.*

Exercise 46. *

1. *Describe a group presentation of* $\overset{k}{\underset{i=1}{\#}} \mathbb{T}^2_i$.

2. *Describe a group presentations of* $\overset{k}{\underset{i=1}{\#}} \mathbb{R}\mathrm{P}^2_i$.

Exercise 47. * *Explicitly determine, using the Classification of Finitely Generated Abelian Groups, what the abelianizations of the fundamental groups found in the previous exercises are. What, if anything, distinguishes orientable from non-orientable surfaces? Are any of these abelianized groups isomorphic? Is this invariant (the abelianized fundamental group) a complete invariant for closed surfaces—*i.e.*, is it sufficient to distinguish between any two surfaces?*

Exercise 48. *Suppose that* $M^2 = \mathbb{T}_1 \# \mathbb{T}_2$ *where* $\mathbb{T}_1$ *and* $\mathbb{T}_2$ *are tori and* $M^2 = U \cup V$ *where* U *is an open set of* $\mathbb{T}_1$ *homeomorphic to* T_1-*(a disk),* V *is an open set of* $\mathbb{T}_2$ *homeomorphic to* T_2-*(a disk), and* $U \cap V$ *is homeomorphic to an open annulus. Let* $p \in U \cap V$*. We know from a previous exercise that* $\pi_1(U,p)$ *is generated by two loops* α *and* β*. Likewise,* $\pi_1(V,p)$ *is generated by two loops* γ *and* δ*. Consider the loop* μ *that generates* $\pi_1(U \cap V, p)$*. Represent* μ *in terms of the generators of* $\pi_1(U,p)$*. Now represent* μ *in terms of the generators of* $\pi_1(V,p)$*. So the single loop* μ *is equivalent to two different loops in* M^2*.* $\pi_1(M^2,p)$ *is generated by* $\{\alpha,\beta,\gamma,\delta\}$*. What relations exist among these generators? Give a presentation of* $\pi_1(M^2,p)$ *whose generators are* $\{\alpha,\beta,\gamma,\delta\}$*.*

3.5 Van Kampen's Theorem, II

***Theorem 49** (Van Kampen's Theorem; group presentations version)**.** *Let* $X = U \cup V$*, where* U, V *are open and path connected and* $U \cap V$ *is path connected and non-empty. Let* $x \in U \cap V$*.*

Let $\pi_1(U,x) = \langle g_1,\ldots,g_n | r_1,\ldots,r_m\rangle$*,* $\pi_1(V,x) = \langle h_1,\ldots,h_t | s_1,\ldots,s_u\rangle$ *and* $\pi_1(U\cap V,x) = \langle k_1,\ldots,k_v | t_1,\ldots,t_w\rangle$ *then*

$$\begin{aligned}\pi_1(X,x) \;&= \langle g_1,\ldots,g_n,h_1,\ldots,h_t \mid \; r_1,\ldots,r_m,s_1,\ldots,s_u,\\ &\qquad i_*(k_1) = j*(k_1),\ldots,i_*(k_v) = j*(k_v)\rangle\end{aligned}$$

where i*,* j *are the inclusion maps of* $U \cap V$ *into* U *and* V *respectively.*

Without the language of group presentations, Van Kampen's Theorem is stated as follows:

***Theorem 50** (Van Kampen's Theorem)**.** *Let* $X = U \cup V$ *where* U, V *are open and path connected and* $U \cap V$ *is path connected and non-empty. Let* $x \in U \cap V$*. Then*

$$\pi_1(X, x) \cong \frac{\pi_1(U, x) * \pi_1(V, x)}{N}$$

where N *is the smallest normal subgroup containing* $\{i_*(\alpha) j_*(\alpha^{-1})\}_{\alpha \in \pi_1(U \cap V, x)}$ *and* i, j *are the inclusion maps of* $U \cap V$ *in* U *and* V *respectively. Note that* N *is the set of products of conjugates of* $i_*(\alpha) j_*(\alpha^{-1})$*.*

Exercise 51. ** Use Van Kampen's theorem to explicitly calculate the group presentation of the double torus* $\mathbb{T}^2 \# \mathbb{T}^2$*.*