Chapter 15 Solutions

1) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^2+y^2}$

If $x=0$

$$\lim_{(0,y)\to(0,0)} \frac{0}{0+y^2} = 0$$

If $y=0$

$$\lim_{(x,0)\to(0,0)} \frac{x^2}{x^2+0} = \lim_{x\to 0} \frac{\cancel{x^2}}{\cancel{x^2}} = 1.$$

So $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^2+y^2}$ DNE

2) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy\cos y}{3x^2+y^2}$

If $x=0$,

$$\lim_{(x,y)\to(0,0)} \frac{0}{0+y^2} = 0$$

If $y=0$

$$\lim_{(x,y)\to(0,0)} \frac{0}{3x^2+0} = 0$$

If $x=y$

$$\lim_{(x,x)\to(0,0)} \frac{x^2\cos x}{3x^2+x^2} = \lim_{(x,x)\to(0,0)} \frac{\cancelto{1}{x^2}\cos x}{4\cancelto{1}{x^2}} = \lim_{x\to 0} \frac{\cos x}{4} = \frac{1}{4}$$

So $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy\cos y}{3x^2+y^2}$ DNE

3) $\lim_{(x,y)\to(0,0)} \frac{x^2+3xy+2y^2}{x+y} = \lim_{(x,y)\to(0,0)} \frac{\cancel{(x+y)}(x+2y)}{\cancel{(x+y)}} = \lim_{(x,y)\to(0,0)} \frac{x+2y}{1}$

$= 0.$

So $\lim_{(x,y)\to(0,0)} \frac{x^2+3xy+2y^2}{x+y} = 0.$

4) $\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$

If $x=0$

$\lim_{(0,y)\to(0,0)} \frac{0}{0+y^2} = 0$

If $y=0$

$\lim_{(x,0)\to(0,0)} \frac{2x^2(0)}{x^4+0} = 0$

If $y=mx$ $m \neq 0$

$\lim_{(x,mx)} \frac{2x^2mx}{x^4+m^2x^2} = \lim_{x\to 0} \frac{2x^3m}{x^2(x^2+m^2)} = \lim_{x\to 0} \frac{2mx}{x^2+m^2} = 0$

If $y=x^2$

$\lim_{(x,x^2)\to(0,0)} \frac{2x^2(x^2)}{x^4+x^4} = \lim_{x\to 0} \frac{2x^4}{2x^4} = 1$

Chapter 15 Review Solutions

5) $f(x,y) = \cos(x+y)$

$f_x = -\sin(x+y)(1)$

$f_y = -\sin(x+y)(1)$

Equation for tangent plane

$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

If $x_0 = 0$, $y = 0$

$z_0 = 1$

$z - 1 = -\sin(0)(x-0) + (-\sin(0))(y-0)$

$z - 1 = 0$

If $x_0 = \frac{\pi}{2}$, $y_0 = 0$

$z_0 = f(\frac{\pi}{2}, 0) = 0$

$z - 0 = f_x(\frac{\pi}{2}, 0)(x - \frac{\pi}{2}) + f_y(\frac{\pi}{2}, 0)(y-0)$

$z = -1(x - \frac{\pi}{2}) - 1(y)$

6) $f(x,y) = e^{x+2y}$

$f_x = e^{x+2y}(1)$

$f_y = e^{x+2y}(2)$

$L(x,y) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + z_0$

let $x_0 = -2$, $y_0 = 1$ $\quad z_0 = f(x_0,y_0) = 1$

$f_x(-2,1) = e^{-2+2} = 1$

$f_y(-2,1) = e^{-2+2}(2) = 2$

$L(-2.01, .99) = 1(-2.01+2) + 2(.99-1) + 1$

$= -.01 + (-.02) + 1 = .97$

7) find $\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} +$$

$$\frac{\partial f}{\partial x} = -\sin(3x+4y)(3) + 10y$$

$$\frac{\partial f}{\partial y} = -\sin(3x+4y)(4) + 10x$$

$$\frac{\partial x}{\partial s} = 1 \qquad \frac{\partial y}{\partial s} = 4s+0$$

$$\frac{\partial x}{\partial t} = 3 \qquad \frac{\partial y}{\partial t} = 21t^2$$

So

$$\frac{\partial f}{\partial s} = (-\sin(3x+4y)(3) + 10y)(1) + (-\sin(3x+4y)4 + 10x)\,4s$$

$$\frac{\partial f}{\partial t} = (-\sin(3x+4y)(3) + 10y)(3) + (-\sin(3x+4y)4 + 10x)(21t^2)$$

8) $g(x,y,z) = e^{x^2+y^2+z}$

$$\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial g}{\partial z}\frac{dz}{dt}$$

$$\frac{\partial g}{\partial x} = e^{x^2+y^2+z}(2x) \qquad \frac{\partial g}{\partial y} = e^{x^2+y^2+z}(2y)$$

$$\frac{\partial g}{\partial z} = e^{x^2+y^2+z}(1)$$

$$\frac{dx}{dt} = 3 \qquad \frac{dy}{dt} = -1 \qquad \frac{dz}{dt} = 2t-1$$

$$\frac{dg}{dt} = e^{x^2+y^2+z}(2x)(3) + e^{x^2+y^2+z}(2y)(-1) + e^{x^2+y^2+z}(1)(2t-1)$$

9) Find $\frac{\partial z}{\partial x}$

$$2z\cdot\frac{\partial z}{\partial x} + 0 + 2x = 4 \Rightarrow 2z\frac{\partial z}{\partial x} = 4-2x$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{4-2x}{2z} = \frac{2-x}{z}$$

$\frac{\partial z}{\partial y}$)

$$2z\frac{\partial z}{\partial y} + 2y + 0 = 0 \Rightarrow \frac{\partial z}{\partial y} = \frac{-y}{z}$$

$9\frac{1}{2}$) First find a unit vector in the direction of $\langle \frac{1}{2}, 1, 2 \rangle$

$$|\langle \tfrac{1}{2}, 1, 2 \rangle| = \sqrt{\tfrac{1}{4} + 1 + 4} = \frac{\sqrt{21}}{2} \qquad \vec{u} = \langle \tfrac{1}{2}, 1, 2 \rangle \cdot \frac{2}{\sqrt{21}}$$

$$\nabla g = \langle e^{x^2+y^2+z}(2x),\ e^{x^2+y^2+z}(2y),\ e^{x^2+y^2+z}(1) \rangle$$

$$\nabla g(0,1,0) = \langle 0, 2e, e \rangle$$

$$D_{\vec{u}}\, g\,(0,1,0) = \langle 0, 2e, e \rangle \cdot \langle \tfrac{1}{2}, 1, 2 \rangle \cdot \frac{2}{\sqrt{21}}$$

$$= (2e + 2e)\frac{2}{\sqrt{21}} = \frac{8e}{\sqrt{21}}$$

10) Find the min/max of $f(x,y)$

$f_x = 2x + y + 1$

$f_y = x - y^2 + 1$

We need First find critical points

$0 = 2x + y - 1 \Rightarrow x = \frac{-1-y}{2}$

$0 = x - y^2 + 1$

So

$0 = \frac{-1-y}{2} - y^2 + 1 \Rightarrow 0 = -1 - y - 2y^2 + 2$

$\Rightarrow 2y^2 + y - 1 = 0$

$(2y-1)(y+1) = 0$

So $y = \cancel{0}$ and $y = \frac{1}{2}$

If $y = -1$

$x = \frac{-1+1}{2} = 0$

So $(0,-1)$ is a critical point

If $y = \frac{1}{2}$

$x = \frac{-1-\frac{1}{2}}{2} = \frac{-3}{4}$

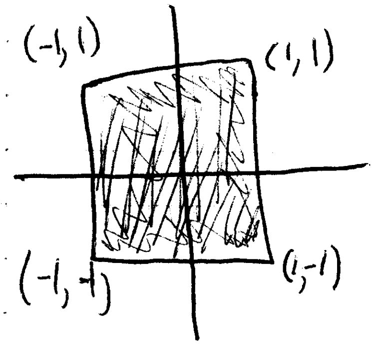
So $(\frac{-3}{4}, \frac{1}{2})$ a critical point

Both are in the region we care about so consider

$f(0,-1) = 0 + 0 + 0 + \frac{1}{3} - 1 + 5 = 4\frac{1}{3} = \frac{13}{3}$

$f(\frac{-3}{4}, \frac{1}{2}) = 4\frac{43}{48} = \frac{235}{48}$

10 continued) Also look at the boundary of the region



If $x=1$ the function is

$$f(1,y) = 1+y+1-\frac{y^3}{3}+y+5 = 7+2y-\frac{y^3}{3}$$

if $x=-1$, we get

$$f(-1,y) = 1-y-1-\frac{y^3}{3}+y+5 = 5-\frac{y^3}{3}$$

If $y=1$, we get

$$f(x,1) = x^2+x+x-\frac{1}{3}+1+5 = x^2+2x+5\tfrac{2}{3}$$

If $y=-1$, we get

$$f(x,-1) = x^2-x+x+\frac{1}{3}-1+5 = x^2+4\tfrac{1}{3}$$

Look at:

$$g_1(y) = f(1,y) = 7+2y-\frac{y^3}{3}$$

$$\frac{dg_1(y)}{dy} = 2-y^2 \qquad y=\pm\sqrt{2} \qquad \sqrt{2}>1 \quad -\sqrt{2}<1$$

no need to compute

$$f(1,\sqrt{2}) = 7+2\sqrt{2}-\frac{2\sqrt{2}}{3} = 7+\frac{4\sqrt{2}}{3} \approx 7+\frac{4(1.4)}{3} \approx 8.8$$

$$f(1,-\sqrt{2}) = 7-2\sqrt{2}+\frac{2\sqrt{2}}{3} = 7-\frac{4\sqrt{2}}{3} \approx 5.2$$

Now look at:

$$g_2(y) = f(-1,y) = 5 - \frac{y^3}{3}$$

$$\frac{dg_2}{dy} = -y^2$$

$$\frac{dg_2}{dy} = 0 \text{ if } y = 0$$

$$g_2(0) = f(-1,0) = 4\tfrac{2}{3} = \frac{14}{3}$$ (actually we already computed this)

$$g_3(x) = f(x,1) = x^2 + 2x + 5\tfrac{2}{3}$$

$$\frac{dg_3}{dx} = 2x + 2$$

$$\frac{dg_3}{dx} = 0 \text{ if } x = -1$$

So need to check $f(-1,1) = 4\frac{2}{3}$

$$g_4(x) = f(x,-1) = x^2 + 4\tfrac{1}{3}$$

$$\frac{dg_4}{dx} = 2x$$

$$\frac{dg_4}{dx} = 0 \text{ if } x = 0$$

$$f(0,-1) = 4\tfrac{1}{3}$$

10 continued)

finally check the corners,

$f(1,1) = 8\frac{2}{3}$

$f(1,-1) = 5\frac{1}{3}$

$f(-1,1) = 4\frac{2}{3}$

$f(-1,-1) = 5\frac{1}{3}$

So the min is $4\frac{1}{3}$ ~~are~~ realized at $(0,-1)$
and the max is $8\frac{2}{3}$ realized at $(+1,+1)$

11) Find min/max of $f(x,y) = 3x^2+y^2+5$
on ~~circle~~ of radius 2 center centered at $(0,1)$

First lets describe the circle

$$(x-0)^2 + (y-1)^2 = 2^2$$
$$x^2 + (y-1)^2 = 4$$

We need La grange here:

$$\nabla f = \langle 6x, 2y \rangle$$

$$\nabla g = \langle 2x, 2y-2 \rangle$$

11 continued)

$6x = 2x\lambda \Rightarrow 2x(3-\lambda) = 0$

$2y = (2y-2)\lambda$

So either $x=0$ or $\lambda = 3$

If $x=0$, $(y-1)^2 = 4$

$y=3$ or $y=-1$

If $\lambda = 3$

$2y = (2y-2)3$

$6 = 4y$

$\frac{3}{2} = y$

So

$x^2 + (\frac{3}{2}-1)^2 = 4$

$x^2 = \frac{15}{4}$

$x = \pm\frac{\sqrt{15}}{2}$

So $(0,3)$, $(0,-1)$, $(\frac{\sqrt{15}}{2}, \frac{3}{2})$, $(\frac{-\sqrt{15}}{2}, \frac{3}{2})$

$f(0,3) = 14$ $\quad$ $f(0,-1) = 6$

$f(\frac{\sqrt{15}}{2}, \frac{3}{2}) = 3\frac{15}{4} + \frac{9}{4} + 5 = \frac{45+9}{4} + 5 = \frac{27}{2} + 5 = \frac{37}{2}$

$f(\frac{-\sqrt{15}}{2}, \frac{3}{2}) = \frac{37}{2}$ $\quad$ Max at $(\frac{\sqrt{15}}{2}, \frac{3}{2})$, $(\frac{-\sqrt{15}}{2}, \frac{3}{2})$