Chapter 16 · Multiple Integrals

1) $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos x \sin y \, dy\, dx$

$$\int_0^{\frac{\pi}{2}} \cos x \sin y \, dy = \cos x\,(-\cos y)\Big|_0^{\frac{\pi}{2}} = 0 + \cos 0 \cdot \cos x$$

$$= \cos x$$

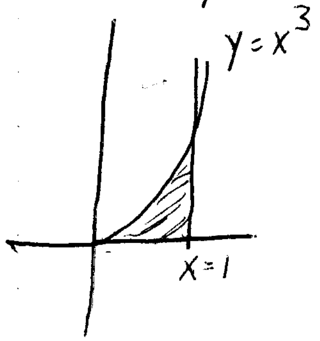
$$\int_0^{\frac{\pi}{2}} \cos x \, dx = \sin x \Big|_0^{\frac{\pi}{2}} = 1 - 0 = 1$$

2) $\displaystyle\int_0^1\int_0^1 e^{2x+y} \, dx\, dy$

$$\int_0^1 e^{2x+y}\, dx = \frac{e^{2x+y}}{2}\Big|_0^1 = \frac{e^{y+2}}{2} - \frac{e^y}{2}$$

$$\int_0^1 \left(\frac{e^{y+2}}{2} - \frac{e^y}{2}\right) dy = \frac{e^{y+2}}{2} - \frac{e^y}{2}\Big]_0^1 = \frac{e^3}{2} - \frac{e}{2} - \frac{e^2}{2} + \frac{1}{2}$$

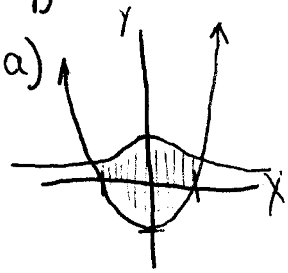
3) $\displaystyle\int_0^1\int_{\sqrt[3]{y}}^1 e^{y/x}\, dx\, dy = \int_0^1\int_0^{x^3} e^{y/x}\, dy\, dx$



$$\int_0^{x^3} e^{y/x}\, dy = x e^{y/x}\Big]_{y=0}^{x^3} = x e^{x^2} - x$$

$$\int_0^1 (x e^{x^2} - x)\, dx = \frac{e^{x^2}}{2} - \frac{x^2}{2}\Big]_0^1 = \frac{e}{2} - \frac{1}{2} - \frac{1}{2} + 0 = \frac{e}{2} - 1$$

4) $\iint_D x\,dA$

a) 

$x^2-1 = \frac{1}{1+x^2}$

$x^4-1=1$

$x^4=2$

$x=\pm\sqrt[4]{2}$

$y=\sqrt{2}-1$

$$\int_{x=-\sqrt[4]{2}}^{\sqrt[4]{2}} \int_{x^2+1}^{\frac{1}{1+x^2}} x\,dy\,dx$$

$$\int_{x^2-1}^{\frac{1}{1+x^2}} x\,dy = xy\Big|_{x^2+1}^{\frac{1}{1+x^2}} = \frac{x}{1+x^2} - x^3 + x$$

$$\int_{-\sqrt[4]{2}}^{\sqrt[4]{2}} \left(\frac{x}{1+x^2} - x^3+x\right)dx = \frac{\ln(1+x^2)}{2} - \frac{x^4}{4} + \frac{x^2}{2}\Bigg]_{x=-\sqrt[4]{2}}^{\sqrt[4]{2}}$$

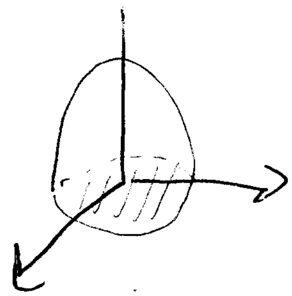
$\cancel{= \frac{\ln 2}{2} - \frac{1}{4} + \frac{1}{2} - 0 - 0 + 0}$

$\cancel{= \frac{\ln 2}{2} + \frac{1}{4}}$

$$= \frac{\ln(1+\sqrt{2})}{2} - \frac{2}{4} + \frac{\sqrt{2}}{2} - \left(\frac{\ln(1+\sqrt{2})}{2} - \frac{(\sqrt[4]{2})^4}{4} + \frac{\sqrt{2}}{2}\right)$$

$$= 0$$

5)

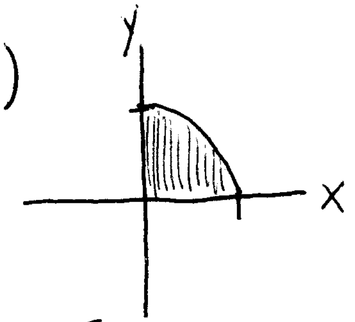


$$\int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} dx\,dy$$

$$= \int_0^{2\pi}\int_0^5 (25-r^2)\,r\,dr\,d\theta = \int_0^{2\pi}\left(\frac{25r^2}{2}-\frac{r^4}{4}\right)\Bigg|_0^5 d\theta$$

$$= 2\pi\left(\frac{625}{2}-\frac{625}{4}\right) = \pi\left(\frac{625}{2}\right)$$

6)



$$\int_0^1\int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{4-x^2-y^2}}\,dy\,dx$$

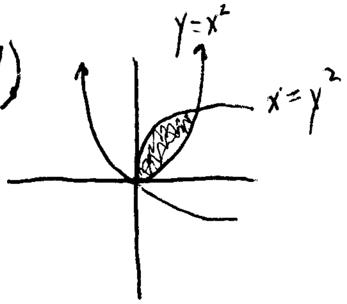
$$\int_0^{\frac{\pi}{2}}\int_{r=0}^1 \frac{1}{\sqrt{4-r^2}}\,r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} -\frac{1}{\sqrt{u}}\,\frac{du}{2}\,d\theta = \int_0^{\frac{\pi}{2}} -u^{\frac{1}{2}}\Big|_{r=0}^1 d\theta$$

$u = 4-r^2$

$du = -2r\,dr$

$-\frac{du}{2} = r\,dr$

$$= \int_0^{\frac{\pi}{2}} -\sqrt{4-r^2}\,\Big|_{r=0}^1 d\theta = \int_0^{\frac{\pi}{2}}\left(-\sqrt{3}+\sqrt{4}\right)d\theta = \frac{\pi}{2}\left(2-\sqrt{3}\right)$$

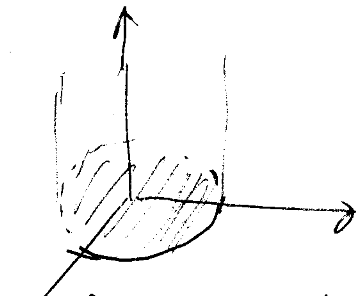
7) 

$$\int_0^1\int_{y=x^2}^{\sqrt{x}} \sqrt{\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2+1}\,dy\,dx$$

$$=\int_0^1\int_{y=x^2}^{\sqrt{x}}\sqrt{4+4+1}\,dy\,dx=\int_0^1\int_{y=x^2}^{\sqrt{x}} 3\,dy\,dx$$

$$=\int_0^1 3y\Big|_{x^2}^{\sqrt{x}}dx=\int_0^1(3\sqrt{x}-3x^2)dx=2x^{\frac{3}{2}}-x^3\Big|_0^1=2-1-0-0$$

$$=1$$

8) 

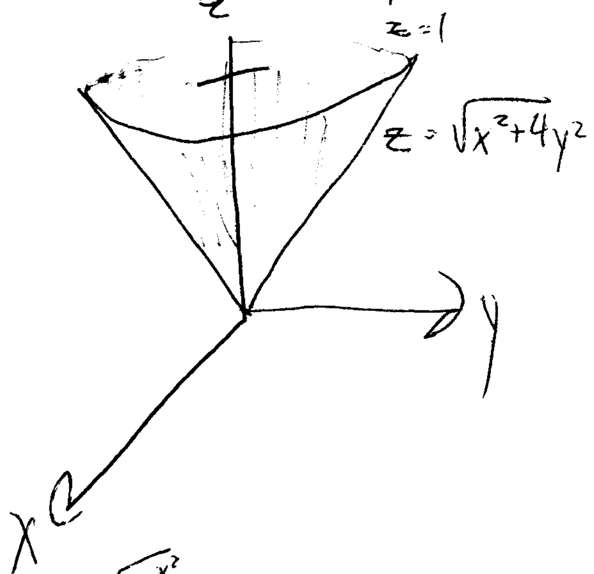
$f_x=3x^2$ $f_y=3y^2$

Domain is a disk of radius 10

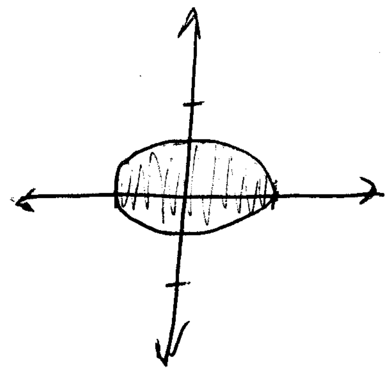
$$\iint_D \sqrt{9x^4+9y^4+1}\,dA$$

That's the best we can do

9) $z = \sqrt{x^2+4y^2}$



$1 = \sqrt{x^2+4y^2} \Rightarrow 1 = x^2+4y^2$



$$V = \int_{y=-1}^{1} \int_{y=-\frac{\sqrt{1-x^2}}{2}}^{\frac{\sqrt{1-x^2}}{2}} \int_{z=\sqrt{x^2+4y^2}}^{1} dV$$

10) Evaluate

$$I = \iiint_R (x+y+z)\,dV = \int_{x=0}^{1} \int_{y=1}^{1} \int_{z=0}^{x+2y} (x+y+z)\,dz\,dy\,dx$$

$$\int_{z=0}^{x+2y} (x+y+z)\,dz = (x+y)z + \frac{z^2}{2}\Big|_0^{x+2y} = (x+y)(x+2y) + \frac{x^2+4xy+4y^2}{2}$$

$$= x^2+2xy+2y^2+\frac{x^2+4xy+4y^2}{2}$$

$$= \frac{3x^2}{2} + 4xy + 4y^2$$

$$I = \int_{x=0}^{1}\int_{y=-1}^{1} \left(\frac{3x^2}{2} + 4xy + 4y^2\right) dy\, dx$$

$$\int_{y=-1}^{1} \left(\frac{3x^2}{2} + 4xy + 4y^2\right) dy = \frac{3x^2}{2}y + 2xy^2 + \frac{4}{3}y^3 \Big|_{y=-1}^{1}$$

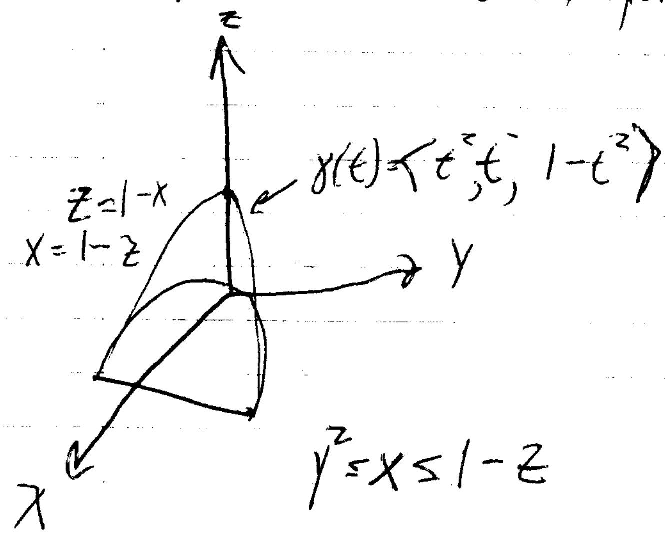
$$= \frac{3x^2}{2} + 2x + \frac{4}{3} - \left(-\frac{3x^2}{2} + 2x - \frac{4}{3}\right)$$

$$= 3x^2 + \frac{8}{3}$$

So

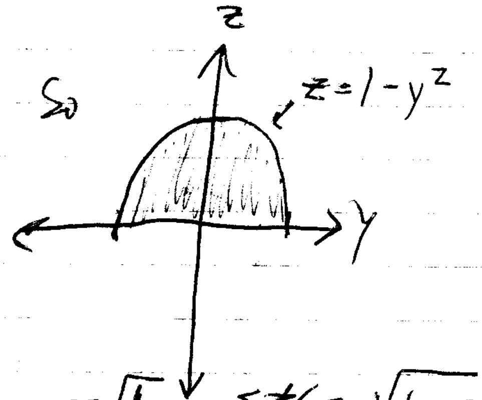
$$I = \int_{x=0}^{1} \left(3x^2 + \frac{8}{3}\right) dx = \cancel{8}\, x^3 + \frac{8}{3}x \Big|_0^1 = 1 + \frac{8}{3} - 0 = \frac{11}{3}$$

Now switch to $dx\,dy\,dz$



$y^2 \le x \le 1 - z$

So



$-\sqrt{1-z} \le \cancel{z} \le \sqrt{1-z}$

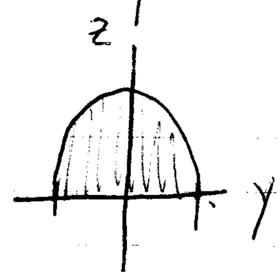
$0 \le z \le 1$

$$V = \int_{z=0}^{1} \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{y^2}^{1-z} dx\,dy\,dz$$

Now switch to $dx\,dz\,dy$

As before

$y^2 \le x \le 1 - z$

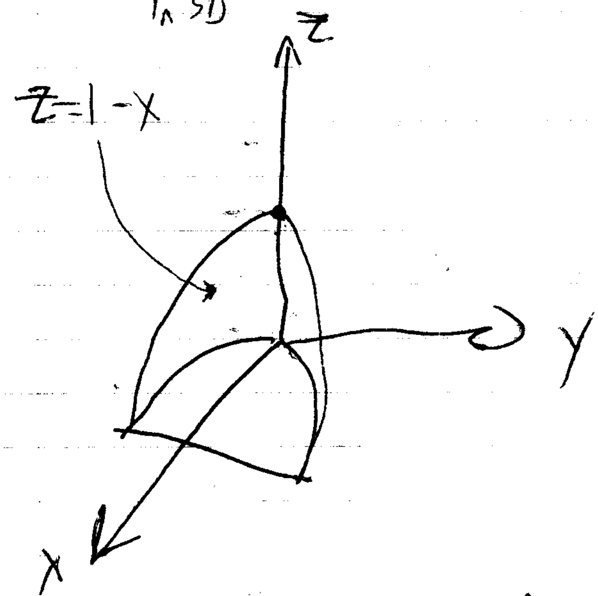


$0 \le z \le 1 - y^2$, $-1 \le y \le 1$

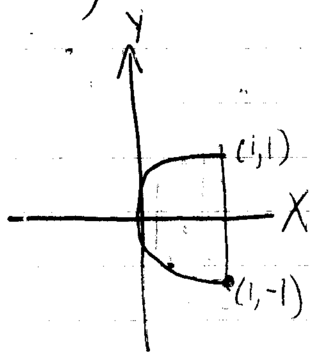
$$\int_{y=-1}^{1} \int_{z=0}^{1-y^2} \int_{x=y^2}^{1-z} dx\,dz\,dy$$

11) $D=\{(x,y,z) \mid 0\le z\le 1-x,\ -1\le y\le 1,\ y^2\le x\le 1\}$

In 3D

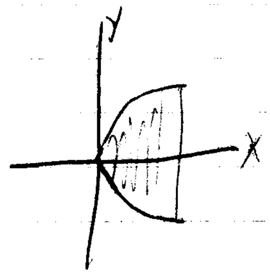


In 2D,



first switch to dzdydx

So $0\le z\le 1-x$ look at



$-\sqrt{x}\le y\le \sqrt{x}$, $0\le x\le 1$

$D=\{(x,y,z) \mid 0\le z\le 1-x,\ 0\le x\le 1,\ -\sqrt{x}\le y\le \sqrt{x}\}$

$$\int_{x=0}^{1}\int_{y=-\sqrt{x}}^{\sqrt{x}}\int_{z=0}^{1-x} dz\,dy\,dx$$