

February 28, 2008
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Exam Review For Second Midterm

1 Practice Exam

1. a) Find the equation of a plane containing the points: $(1, 2, -1)$, $(2, 3, 2)$, $(0, 1, 1)$. [Hint: Find the normal vector this plane, by finding two vectors in the plane and taking their cross-product.]

Call $p_1 = (1, 2, -1)$, $p_2 = (2, 3, 2)$, $p_3 = (0, 1, 1)$.

Call the \vec{v}_{12} the vector that goes from p_1 to p_2 . Call the \vec{v}_{32} the vector that goes from p_3 to p_2 .

$$\vec{v}_{12} = \langle 2 - 1, 3 - 2, 2 - (-1) \rangle = \langle 1, 1, 3 \rangle \quad \vec{v}_{32} = \langle 2 - 0, 3 - 1, 2 - 1 \rangle = \langle 2, 2, 1 \rangle$$

So,

$$\vec{v}_{12} \times \vec{v}_{32} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 3 \\ 2 & 2 & 1 \end{vmatrix} = \hat{i} \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = -5\hat{i} + 5\hat{j} + 0\hat{k}$$

So, $-5\hat{i} + 5\hat{j} + 0\hat{k}$ is the normal vector for our plane and $-5x + 5y + 0z + d = 0$ is the equation for our plane. Since $(1, 2, -1)$ is a solution for the equation of the plane, $-5 \cdot 1 + 5 \cdot 2 + 0 + d = 0$. Hence, $d = 5$.

So, $-5x + 5y + 5 = 0$ or $-x + y + 1 = 0$ are solutions for the plane.

b) Describe the intersection of the planes:

i) $3x + 4y - 2z + 2 = 0$, $-3x - 4y + 2z - 8 = 0$

The normal vectors to the planes $\langle 3, 4, -2 \rangle$ and $\langle -3, -4, 2 \rangle$ are scalar multiples of one another. So the planes are parallel and have no intersection.

ii) $x + y + z + 4 = 0$, $3x - y + 2z = 0$

The intersection of these two planes is a line contained in both planes. The direction vector of this line is perpendicular to both normal vectors. So, we want to find a vector perpendicular to two given vectors $\langle 1, 1, 1 \rangle$ and $\langle 3, -1, 2 \rangle$. We know that the cross product of these two vectors is a vector perpendicular to both of these vectors.

$$\langle 1, 1, 1 \rangle \times \langle 3, -1, 2 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 3 & -1 & 2 \end{vmatrix} = \hat{i} \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = 3\hat{i} + (1)\hat{j} + (-4)\hat{k}$$

So the line shared by the two planes has direction $\langle 3, 1, -4 \rangle$. (This is a good time to check that this vector is indeed perpendicular to both of the normal vectors. Seeing that the dot product with one of the normal vectors is zero usually catches any errors.)

To find a point on the line shared by these two planes, we need to solve both equations for the planes simultaneously. When $x=0$, the equations for the planes are

$$\begin{aligned}y + z + 4 &= 0 \\ -y + 2z &= 0\end{aligned}$$

After adding these equations together, $3z + 4 = 0$ and $z = -4/3$. $y = -8/3$. So the point $(0, -8/3, -4/3)$. From here we can get the symmetric equations for the line shared by both planes to be:

$$\frac{x}{3} = \frac{y + \frac{8}{3}}{-1} = \frac{z + \frac{4}{3}}{-2}$$

.

c) Find the angle between the planes: $x+2y+5z=0$, $4x+3y+z+4=0$.

The angle between these two planes is just the angle between the normal vectors of these two planes, $\langle 1, 2, 5 \rangle, \langle 4, 3, 1 \rangle$. We use the formula: Notice, $|\langle 1, 2, 5 \rangle| = \sqrt{1+4+25} = \sqrt{30}$
 $|\langle 4, 3, 1 \rangle| = \sqrt{16+9+1} = \sqrt{26}$

$$\langle 1, 2, 5 \rangle \cdot \langle 4, 3, 1 \rangle = 4 + 6 + 5 = 15$$

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \text{ and get } \cos \theta = \frac{15}{\sqrt{30}\sqrt{26}}.$$

$$\text{So, } \theta = \cos^{-1} \frac{\sqrt{15}}{2\sqrt{13}}.$$

2. a) A line goes through the points $(1, 2, 0)$ and $(2, 3, 1)$, give parametric and symmetric equations for this line. (There are several solutions for the parametric equations. How can they differ?)

First we want to get the direction vector of this line. This is $\langle 2 - 1, 3 - 2, 1 - 0 \rangle = \langle 1, 1, 1 \rangle$.

The symmetric equations for this line are :

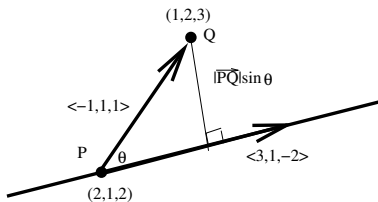
$$x - 1 = y - 2 = z$$

.

If we consider $(1, 2, 0)$ to be at time 0 and $(2, 3, 1)$ at time 1. The parametric equations for the line are $x = 1 + t, y = 2 + t, z = t$. (Notice this depends on what we choose to be the points at time 1 and time 0.)

b) Find the distance from $(1, 2, 3)$ to the line $\frac{x-2}{3} = y - 1 = \frac{z-2}{2}$

If we pick any point on the line, we can create a right triangle that has one leg along the line. Pick the point (2,1,2). Let's call it P and the other point (1,2,3) Q. To get from (2,1,2) to (1,2,3) we would need to travel along the vector $\vec{PQ} = \langle 1 - 2, 2 - 1, 3 - 2 \rangle = \langle -1, 1, 1 \rangle$. The direction vector along the line is $\vec{v} = \langle 3, 1, -2 \rangle$. So these vectors form an angle θ .



We want to find $|\vec{PQ}| \sin \theta$. Using the formula from the cross product, $|\vec{PQ} \times \vec{v}| = |\vec{PQ}| |\vec{v}| \sin \theta$. Actually, moving things around here we see that $\frac{|\vec{PQ} \times \vec{v}|}{|\vec{v}|} = |\vec{PQ}| \sin \theta$

So we need to find $|\vec{v}|$ and $|\vec{PQ} \times \vec{v}|$.

$$|\vec{v}| = \sqrt{3^2 + 1^2 + (-2)^2} = \sqrt{14}$$

$$\vec{PQ} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ 3 & 1 & -2 \end{vmatrix} = \hat{i} \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} - \hat{j} \begin{vmatrix} -1 & 1 \\ 3 & -2 \end{vmatrix} + \hat{k} \begin{vmatrix} -1 & 1 \\ 3 & 1 \end{vmatrix} = -3\hat{i} + 1\hat{j} + (-4)\hat{k}$$

(You might want to check you answer with the dot product here.)

$$|\vec{PQ} \times \vec{v}| = \sqrt{(-3)^2 + 1^2 + (-4)^2} = \sqrt{9 + 1 + 16} = \sqrt{26}$$

$$\text{So the distance is } |\vec{PQ}| \sin \theta = \frac{|\vec{PQ} \times \vec{v}|}{|\vec{v}|} = \frac{\sqrt{26}}{\sqrt{14}} = \frac{\sqrt{13}}{\sqrt{7}}$$

c) If a line goes through the point (-1, 3, -1) and has direction $\langle 2, 3, 1 \rangle$, what are parametric equations that describe this line?

$$\begin{aligned} x &= -1 + 2t \\ y &= 3 + 3t \\ z &= -1 + t \end{aligned}$$

d) To get the equation for a plane we need a normal vector and a point in the plane. We have three points to choose from (-1, 3, -1), (1, 2, 0) and (2, 3, 1) so let's focus on finding a normal vector for the plane.

The direction vector of the line in part a) is $\langle 1, 1, 1 \rangle$ and the direction vector of the line in part b) is $\langle 2, 3, 1 \rangle$. Both are in the plane.

So we want a vector that is perpendicular to both $\langle 2, 3, 1 \rangle$ and $\langle 1, 1, 1 \rangle$.

$$\langle 1, 1, 1 \rangle \times \langle 2, 3, 1 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{vmatrix} = \hat{i} \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -2\hat{i} - (-1)\hat{j} + (1)\hat{k}$$

So the normal vector to our plane is $\langle -2, 1, 1 \rangle$. Now let's choose $(1, 2, 0)$ as our point. So the equation is:

$$-2(x - 1) + 1(y - 2) + 1(z - 0) = 0.$$

e) In order to get a line perpendicular to the plane, we need only pick a line that is in the direction of our normal vector and goes through any point in space. (This means there are going to be lots of solutions.) My favorite point to pick is the point $(0,0,0)$. So the my line perpendicular to the plane is going to be described by the equations:

$$x = -2t$$

$$y = t$$

$$z = t$$

.

3. a) Find unit vectors in the direction of $\vec{a} = \langle 2, 0, -4 \rangle$ and $\vec{b} = \langle 3, 1, 1 \rangle$.

$$\text{Here, we want a vector } \vec{u}_1 = \frac{\vec{a}}{|\vec{a}|} = \frac{\langle 2, 0, -4 \rangle}{\sqrt{2^2 + 0^2 + (-4)^2}} = \frac{\langle 2, 0, -4 \rangle}{\sqrt{20}} = \langle \frac{1}{\sqrt{5}}, 0, \frac{-2}{\sqrt{5}} \rangle .$$

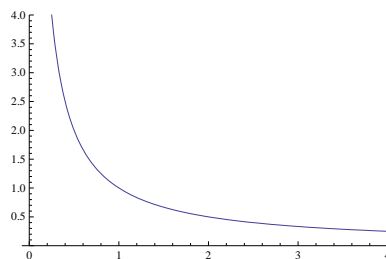
$$\text{Also, we want the vector } \vec{u}_2 = \frac{\vec{b}}{|\vec{b}|} = \frac{\langle 3, 1, 1 \rangle}{\sqrt{3^2 + 1^2 + 1^2}} = \frac{\langle 3, 1, 1 \rangle}{\sqrt{11}} = \langle \frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}} \rangle .$$

b) Are these vectors perpendicular? If not find a vector that is perpendicular to \vec{a} .

No, \vec{a} is not perpendicular to \vec{b} since $\vec{a} \cdot \vec{b} = 6 + 0 - 4 = 2 \neq 0$. There are going to be lots of vectors perpendicular to \vec{a} the trick is to pick one. $\vec{a} \times \vec{b}$ is a good choice. $\vec{a} \times \vec{b} = \langle 4, -14, 2 \rangle$.

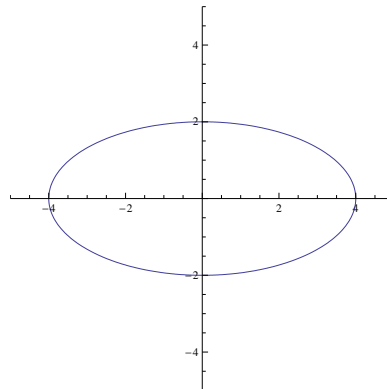
4. a)

i) Notice that $x = t^2$ and $y = 1/t^2$ so $y = 1/x$ and $x > 0$.



So our graph is just:

ii) We can see that $x^2/16 + y^2/4 = 1$. So this is an ellipse.



So our graph is just:

b) i)
 $\frac{dx}{dt} = 2t$
 $\frac{dy}{dt} = \frac{-2}{t^3}$

Notice, $\frac{dx}{dt} = 2t$ is only 0 at $t = 0$, but this is not in the domain. So, the tangent line can never be vertical.

Also, $\frac{dy}{dt} = \frac{-2}{t^3}$ is never 0 so the tangent line can never be horizontal.

ii)
For $\frac{dx}{dt} = 4 \cos t$, this is 0 at $t = \pi/2, 3\pi/2, 5\pi/2, \dots$

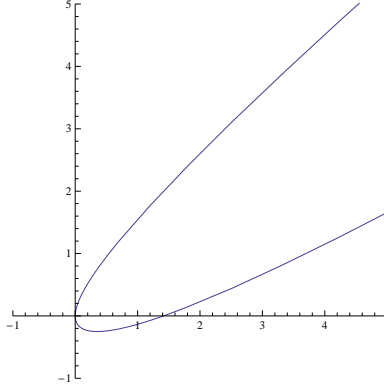
For $\frac{dy}{dt} = -2 \sin t$, this is 0 at $t = 0, \pi, 2\pi, \dots$

Since there aren't values for t that make both zero at the same time. The tangent line is horizontal at $t = 0, \pi, 2\pi, \dots$ and vertical at $t = \pi/2, 3\pi/2, 5\pi/2, \dots$

c) We need to use the arc length formula here.

$$\text{Arc Length} = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

where α and β are values for t .



The trick is to figure out when $y = 0$ because those will give us our limits of integration.

$$y = 0 = t^2 - t = t(t - 1) \text{ so if } t = 0, 1.$$

Now all we need is $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 2t\sqrt{2} \quad \frac{dy}{dt} = 2t - 1$$

$$\text{Arc Length} = \int_0^1 \sqrt{(2t\sqrt{2})^2 + (2t - 1)^2} dt$$

At best you will be able to simplify this to

$$= \int_0^1 \sqrt{8t^2 + 4t^2 + 4t + 1} dt = \int_0^1 \sqrt{12t^2 + 4t + 1} dt$$

d)

To find the area between the curve and the x-axis. We use the formula

$$\text{Area} = \int_{\alpha}^{\beta} y(t)x'(t)dt$$

$$y(t) = t^2 - t$$

$$x'(t) = \frac{dx}{dt} = 2t\sqrt{2}$$

Just as before our limits of integration are 0 and 1.

$$\text{Area} = \int_0^1 (t^2 - t)(2t\sqrt{2})dt = \int_0^1 (2t^3\sqrt{2} - 2t^2\sqrt{2})dt = \left. \frac{\sqrt{2}}{2}t^4 - \frac{2}{3}t^3 \right|_{t=0}^1 = \frac{\sqrt{2}}{2} - \frac{2}{3}.$$