Explanation of 11.6

This sheet serves to better explain the technical points of Theorems 1 and 2 in Section 11.6 in the book. Let's start with what we know.

Theorem 1. *Let F be a fixed point (called the* ***focus****) and l be a fixed line (called the* ***directrix****) in a plane. Let e be a fixed positive number (called the* ***eccentricity****). The set of all points P in the plane such that*

$$\frac{|PF|}{Pl} = e$$

is a conic section. The conic is:

(a) an ellipse if $e < 1$
(b) a parabola if $e = 1$
(c) a hyperbola if $e > 1$

Theorem 2. *A polar equation of the form*

$$r = \frac{ed}{1 \pm e\cos\theta} \text{ or } r = \frac{ed}{1 \pm e\sin\theta}$$

represents a conic section with eccentricity e. The conic is an ellipse if $e < 1$, a parabola if $e = 1$, or a hyperbola if $e > 1$.

Both of these theorems have results that depend on e and give us three cases, the parabola, the hyperbola, and the ellipse. Now, the equations in Theorem 2 has four important pieces of information to fill in: the choice of what e is, the choice $\sin\theta$ or $\cos\theta$, the choice of what d is and the choice of $+$ or $-$.

We at least have some idea of what different choices for e do. The big picture is that e affects the overall shape. There are three cases to consider $e > 1$, $e = 1$, and $e > 1$.

Let's start with $e = 1$, we know we get a parabola and try and write out this equation in polar coordinates. If we choose to the equation,

$$r = \frac{d}{1 \pm \sin\theta}$$

we can clear the denominator to get,

$$r \pm r\sin\theta = d.$$

Using $y = r\sin\theta$, we get

$$r \pm y = d \text{ or } r = \mp y + d.$$

Squaring both sides gives,

$$r^2 = y^2 \mp 2dy + d^2 \text{ or } x^2 + y^2 = y^2 \mp 2dy + d^2$$

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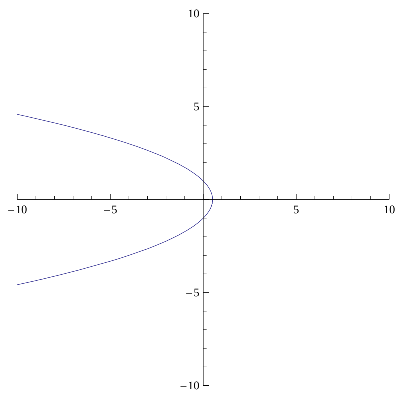

Figure 1: Here $r = \frac{1}{1+\cos\theta}$ so $d = 1$ and we choose +

Which becomes,

$$\frac{\mp x^2}{2d} = y \mp \frac{d}{2}.$$

This is a parabola with vertex $x = 0$ and $y = \pm\frac{d}{2}$.
If there were a $\cos\theta$ in the bottom to start we would get

$$\frac{\mp y^2}{2d} = x \mp \frac{d}{2}.$$

Changes in e Changing e changes the shape of the graph away from a parabola.

Choosing $\sin\theta$ **or** $\cos\theta$ If we choose the sin then we get a vertically oriented parabola and a horizontal directrix ($y = \pm d$). If we choose the cos, we get a horizontally oriented parabola and a horizontal directrix ($x = \pm d$).

Changes in d The choice of affects where the directrix is. This together with the choice of "+" or "-" the directrix is x or y is $\pm d$. Also we can see that d affects where the vertex is and how wide our parabola is.

Choosing "+" or "-"

Ellipses and Hyperbolas Now let's consider what happens when $e \neq 1$. First choose $r = \frac{ed}{1 \pm e\sin\theta}$ (here we are seeing what happens when we choose $\sin\theta$).

So we have

$$r = \frac{ed}{1 \pm e\sin\theta}$$

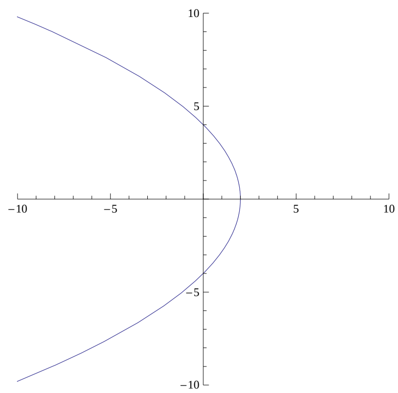

Figure 2: Here $r = \frac{4}{1+\cos\theta}$ so $d = 4$ and we choose +

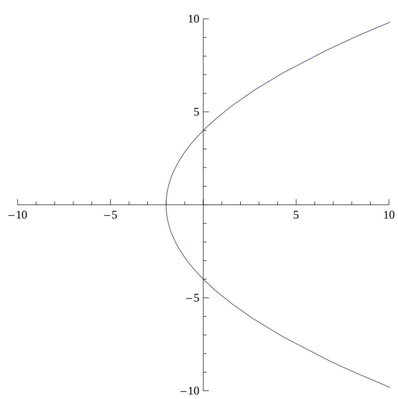

Figure 3: Here $r = \frac{4}{1-\cos\theta}$ so $d = 4$ and we choose -

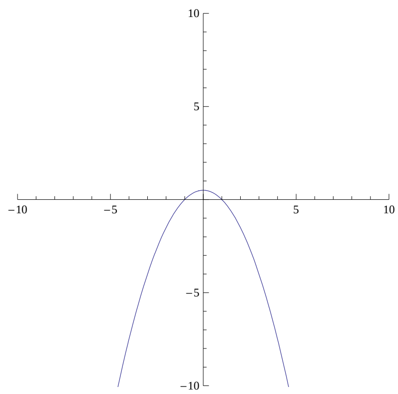

Figure 4: Here $r = \frac{1}{1+\sin\theta}$ so $d = 1$ and we choose $+$

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By clearing out the denominator we get

$$r \pm er\sin\theta = ed$$

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Personally, I have a better feel for these conic sections in cartesian coordinates. So let's try to get rid of r and θ. First use $y = r\sin\theta$.

Now the equation is

$$r \pm ey = ed$$

So we still have to get rid of that r. This next bit is a little tricky, but it will get rid of the r.

First, move y over to the other side of the equation then square both sides,

$$r = \mp ey + ed \text{ this becomes } r^2 = e^2y^2 \mp 2e^2dy + e^2d^2.$$

Now we can use that $r^2 = x^2 + y^2$, and we get that,

$$x^2 + y^2 = e^2y^2 \mp 2e^2dy + e^2d^2$$

Shifting the y's to one side of the equation yields,

$$x^2 + y^2 - e^2y^2 \pm 2e^2dy = e^2d^2$$

In order to make the equation look nice we need to complete the square by adding $\frac{e^2d}{1-e^2}$ to both sides,

$$x^2 + (1-e^2)y^2 \pm 2e^2dy + \frac{e^4d^2}{1-e^2} = e^2d^2 + \frac{e^4d^2}{1-e^2}$$

Now, we can factor out a $1-e^2$ from each term with a y in it and simplify the right hand side to get,

$$x^2+(1-e^2)(y^2\pm\frac{2e^2dy}{1-e^2}+\frac{e^4d^2}{(1-e^2)^2}) = e^2d^2+\frac{e^4d^2}{1-e^2} = (e^2d^2)(1-\frac{e^2}{1-e^2}) = \frac{e^2d^2}{1-e^2}$$

When we factor the left hand side we get,

$$x^2+(1-e^2)(y\pm\frac{e^2d}{1-e^2})^2 = \frac{e^2d^2}{1-e^2}$$

In order to get the right hand side to be 1, we multiply by the inverse of the right hand side which gives us,

$$\frac{1-e^2}{e^2d^2}x^2+\frac{(1-e^2)^2}{e^2d^2}(y\pm\frac{e^2d}{1-e^2})^2 = 1$$

If we had started with $r=\frac{ed}{1\pm\cos\theta}$ we would get,

$$\frac{(1-e^2)^2}{e^2d^2}(x\pm\frac{e^2d}{1-e^2})^2+\frac{1-e^2}{e^2d^2}y^2 = 1$$

In the case of an ellipse $e<1$ **Changes in e** Affect how big the ellipse is. The closer e is to 1, the bigger it is. Also e affects where the center of the ellipse is.

Choosing $\sin\theta$ **or** $\cos\theta$ This chooses whether the ellipse is longer in the vertical direction (when we choose $\sin\theta$)or the horizontal direction (when we choose $\cos\theta$). Also, the center ($x=0$,$y=\mp\frac{e^2d}{1-e^2}$) of the ellipse lies on the y-axis when we choose sin and the center ($x=\mp\frac{e^2d}{1-e^2}$,$y=0$) lies on the x-axis when we choose cos.

Changes in d Changes in d along with the sign affect where the directrix lies. It also affects the size of the ellipse. The bigger the d the bigger the ellipse.

Choosing "+" or "-" Affects where the center is and really determines which focus of the ellipse the orgin is (in chapter 11.5 we saw an ellipse had two foci). Also affects where the directrix is.

From the book we know that one of the foci of this ellipse is the origin.

The book has examples of changes in e. This will focus on changes in the sign on the bottom

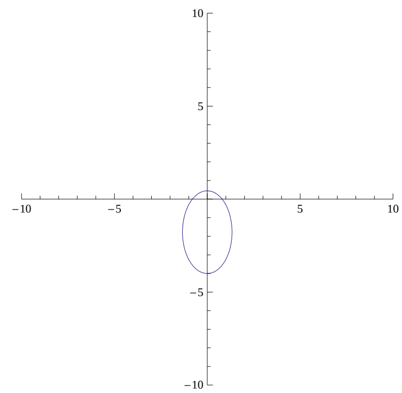

Figure 5: Here $r = \frac{.8}{1+.8\sin\theta}$ so $e = .8$ $d = 1$ and we choose +

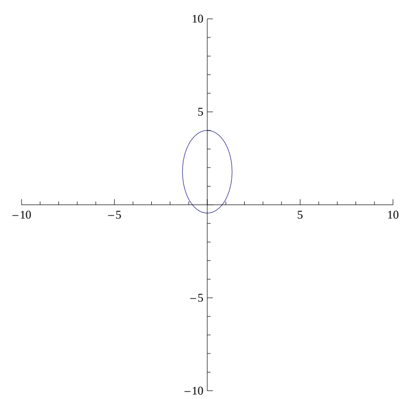

Figure 6: Here $r = \frac{.8}{1-.8\sin\theta}$ so $e = .8$ $d = 1$ and we choose -

Hyperbolas For $e > 1$, we get a hyperbola.

Changes in e The bigger the e the flatter the hyperbola.

Choosing $\sin\theta$ **or** $\cos\theta$ This chooses whether the hyperbola is shaped like sides of an hour glass (when we choose $\sin\theta$)or like a smile and a frown (when we choose $\cos\theta$). Also, the point where the asymptotes cross ($x = 0$,$y = \mp\frac{e^2d}{1-e^2}$) of the hyperbola lies on the y-axis when we choose sin and the crossing point ($x = \mp\frac{e^2d}{1-e^2}$,$y = 0$) lies on the x-axis when we choose cos. This choice also affects the directrix, which should be horizontal when we choose sin and vertical ($x = \pm d$) when we choose cos.

Changes in d Changes in d and Choosing plus or minus affect where the center is. d also affects the directix and the shape of the hyperbola. The higher the d the flattr the hyperbola.

Choosing "+" or "-" This is covered in changes in d.