

Algebra Review

How well do you remember your algebra?

1 The Order of Operations

What do we mean when we write

$$2 \times 3 + 4?$$

If we multiply 2×3 we get 6 and adding 4 gives 10. But, if we add $3 + 4 = 7$ first, then multiply by 2 then we get 14 as the answer. Which answer do we want when we evaluate this?

The point is that *the order we do these operations matters*. We must decide a convention of which operations to do first. This is called **order of operations**.

Here are the conventions we adopt for the order of operations:

- Do things inside parentheses first. Of all parentheses, work on innermost parentheses first.
- Next compute any exponents.
- Now multiply and divide.
- Finally, do any addition or subtraction.

So in the case of the above problem, we get

$$2 \times 3 + 4 = 6 + 4 = 10, \text{ not } 14.$$

Here are a couple of other important notations that we will be using. We often write multiplication with a “.” or sometimes without a symbol at all. All of the following mean “ a times b ”.

$$a \times b = a \cdot b = ab$$

A horizontal line or a “/” means divide.

$$a \div b = a/b = \frac{a}{b}$$

Algebra Hazard # 1: No Division by Zero.
For example, note that $x^2 = 2x \Rightarrow x = 2$ is incorrect.

Note also that in

$$\frac{x^2 + 2}{x - 1}$$

there is an understood parenthesis on the numerator and denominator, so what we mean when we write this is

$$\frac{(x^2 + 2)}{(x - 1)} \text{ or equivalently } (x^2 + 2) \div (x - 1)$$

1.1 Problems

Simplify the following expressions and show each individual step in your work:

Example:

$$\frac{(3 - 1)^2 - 1}{3 \cdot 4 - 7} = \frac{2^2 - 1}{12 - 7} = \frac{4 - 1}{5} = \frac{3}{5}$$

$$\#1. \quad 2(5 - 2)^2 + 3 \quad (\text{answer : } 21)$$

$$\#2. \quad 3^2 + 6 \div 3 - 1 \quad (\text{answer : } 10)$$

$$\#3. \quad \frac{3^2 + 6}{3 - 1} \quad (\text{answer : } \frac{15}{2})$$

2 Associative, Commutative, and Distributive Properties

Multiplication and addition both have the properties of associativity and commutativity.

If the variables a , b , and c all represent numbers, the following are the **associative properties** for addition and multiplication:

$$\text{Addition : } a + (b + c) = (a + b) + c$$

$$\text{Multiplication : } a(bc) = (ab)c$$

This means that we can add or multiply numbers in any order and you will always get the same thing. Here are a couple of examples with real numbers:

$$2 + (4 + 1) = 7 = (2 + 4) + 1$$

$$6 \cdot (2 \cdot 3) = 36 = (6 \cdot 2) \cdot 3$$

The **commutative property** is easy, it is

$$a + b = b + a \quad (\text{addition}) \quad \text{and} \quad ab = ba \quad (\text{multiplication})$$

This means that it doesn't matter which order we write things in when using addition or multiplication. (Here's something to start thinking about – Is the commutative property true for subtraction and division?)

Some examples with numbers:

$$2 + 3 = 5 = 3 + 2 \quad 2 \times 3 = 6 = 3 \times 2$$

The **distributive property** is

$$a(b + c) = ab + ac$$

or from the left

$$(b + c)a = ba + ca$$

This property is a common one people forget, so make sure you understand it.

Here it is with some numbers:

$$4(5 + 1) = 24 = 4 \cdot 5 + 4 \cdot 1$$

The associative, commutative, and distributive properties also apply to subtraction and division, but we have to be careful. To apply the properties to subtraction, we need to first convert the subtraction to addition. Subtraction is the same as adding a negative number. Here is an example of this:

$$3 - 2 = 3 + (-1) \cdot 2 = 3 + (-2)$$

The same is true for more complicated expressions:

$$(x - 2) - (2x - 1) = -(2x - 1) + (x - 2)$$

The distributive property can further be used to simplify the above expression.

$$-(2x - 1) + (x - 2) = (-1)(2x - 1) + (1)(x - 2) = -2x + 1 + x - 2 = -x - 1$$

Similarly, in order to apply the associative, commutative and distributive properties to division, we think of division as multiplication. Dividing is the same as multiplying by the reciprocal.

$$a \div b = a \times \frac{1}{b}$$

Here's an example using the distributive property with division.

$$(x^2 - 1) \div 2 = (x^2 + (-1)) \cdot \frac{1}{2} = x^2 \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2}$$

The distributive property is also where the whole business of “factoring” comes from. Consider

$$2x + 4y$$

Both terms in the sum have a factor of two in them, so we can “factor it out” or use the distributive property to write it a different way.

$$2x + 4y = 2 \cdot x + 2 \cdot 2y = 2(x + 2y)$$

We will do this sort of thing a lot in this class.

Algebra Hazard # 2: Fake Distributive Properties.
 Note that $(x + 2)^2 \neq x^2 + 2^2$ and $\sqrt{x + 2} \neq \sqrt{x} + \sqrt{2}$

You may have learned the FOIL (First Outside Inside Last) method for multiplying things like $(x + 2)(x - 3)$. It comes from the distributive property. Just treat the stuff in the first set of parentheses as a single number and distribute it into the second set of parentheses.

$$(x + 2)(x - 3) = (x + 2)x + (x + 2)(-3)$$

Now use the distributive property again, this time from the left. Finally, we simplify by combining like terms.

$$\dots = x \cdot x + 2x + x(-3) + 2(-3) = x^2 + 2x - 3x - 6 = x^2 - x - 6$$

Notice that what we have done is exactly the FOIL method.

We can use this same approach to multiply bigger things.

To multiply $(x^2 + x - 2)(x^2 - 2x + 1)$, first distribute $(x^2 + x - 2)$ into $(x^2 - 2x + 1)$.

$$\begin{aligned} (x^2 + x - 2)(x^2 - 2x + 1) &= (x^2 + x - 2) \cdot x^2 + (x^2 + x - 2)(-2x) + (x^2 + x - 2)(1) \\ &= x^2(x^2 + x - 2) + (-2x)(x^2 + x - 2) + (1)(x^2 + x - 2) \end{aligned}$$

Now distribute again.

$$\begin{aligned} &= x^2 \cdot x^2 + x^2 \cdot x + x^2(-2) + (-2x)x^2 + (-2x)x + (-2x)(-2) + x^2 + x - 2 \\ &= x^4 + x^3 - 2x^2 - 2x^3 - 2x^2 + 4x + x^2 + x - 2 \\ &= x^4 - x^3 - 3x^2 + 5x - 2 \end{aligned}$$

2.1 Problems

Use the associative, commutative and distributive properties to simplify the following.

#4. $(3a - b) - [2a - (a + b)]$ (ans : $2a$)

#5. $[(a + 3b) - a] - [a - (a - 3b)]$ (ans : 0)

#6. $a - \{2a - [b - (3a - 2b)]\}$ (ans : $-4a + 3b$)

Factor out any factors common to all the terms using the distributive property.

Example : $6x^2 - 4x = 2x(3x - 2)$

#7. $12x - 8y + 30$

#8. $8x^2 - 12x^3y - 28x^4z$

#9. $9abc + 3a^2b^2c^2$

Multiply and simplify by using the distributive property.

$$\#10. \quad (2x - 3)(x + 2)$$

$$\#11. \quad (x - 1)(x^2 - 3x + 2)$$

3 Fractions

Multiplying fractions. This is the easiest, just multiply the numerators(top) together and multiply the denominators(bottom) together.

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

Here's an example with numbers.

$$\frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$$

And another.

$$12 \cdot \frac{1}{3} = \frac{12}{1} \cdot \frac{1}{3} = \frac{12}{3} = \frac{3 \cdot 4}{3 \cdot 1} = \frac{3}{3} \cdot \frac{4}{1} = 1 \cdot 4 = 4$$

This is good, since half of three quarters is three eighths and a third of twelve is four. Also notice that we have used the fact that $12 = \frac{12}{1}$ and that $\frac{3}{3} = 1$. These are important observations that we use often.

To add fractions, we need a common denominator. This amounts to saying that if we want to add things up, it is easier if all the pieces are the same size.

Say we want to add three quarters to one half. The pieces are different sizes, but we know that one half is the same as two quarters. Therefore we can add three quarters plus two quarters. It is easy now, the answer is five quarters. This is common denominators.

Now in more mathematical terms.

$$\frac{3}{4} + \frac{1}{2}$$

Generally, the easiest common denominator to work with is the *least common multiple* of the denominators in question. In this case, the least common multiple of 4 and 2 is 4. Our goal is to write both fractions with denominator 4. The first is done, the second we do as follows, using the fact that $\frac{2}{2} = 1$.

$$\frac{1}{2} = 1 \cdot \frac{1}{2} = \frac{2}{2} \cdot \frac{1}{2} = \frac{2 \cdot 1}{2 \cdot 2} = \frac{2}{4}$$

Now we have

$$\frac{3}{4} + \frac{1}{2} = \frac{3}{4} + \frac{2}{4} = \frac{5}{4}$$

It works the same if the fractions are more complicated. Try

$$\frac{x}{x+1} + \frac{2}{x+2}$$

The least common multiple of the denominator is $(x + 1)(x + 2)$. So this time we need to fix both denominators. First, to fix $\frac{x}{x+1}$ we use the fact that $\frac{x+2}{x+2} = 1$.

$$\frac{x}{x+1} = 1 \cdot \frac{x}{x+1} = \frac{x+2}{x+2} \cdot \frac{x}{x+1} = \frac{x(x+2)}{(x+1)(x+2)}$$

Now to fix $\frac{2}{x+2}$ we use the fact that $\frac{x+1}{x+1} = 1$.

$$\frac{2}{x+2} = 1 \cdot \frac{2}{x+2} = \frac{x+1}{x+1} \cdot \frac{2}{x+2} = \frac{2(x+1)}{(x+1)(x+2)}$$

Because

$$\frac{x}{x+1} = \frac{x(x+2)}{(x+1)(x+2)} \quad \text{and} \quad \frac{2}{x+2} = \frac{2(x+1)}{(x+1)(x+2)}$$

we have

$$\frac{x}{x+1} + \frac{2}{x+2} = \frac{x(x+2)}{(x+1)(x+2)} + \frac{2(x+1)}{(x+1)(x+2)}$$

Now we have written the sum in fractions with a common denominator so we can add and simplify to get

$$\begin{aligned} &= \frac{x(x+2) + 2(x+1)}{(x+1)(x+2)} \\ &= \frac{x^2 + 2x + 2x + 2}{(x+1)(x+2)} \\ &= \frac{x^2 + 4x + 2}{x^2 + 3x + 2} \end{aligned}$$

What if you have a fraction divided by a fraction? As you may already know, you multiply the top fraction by the reciprocal of the bottom fraction. (Recall that you get the reciprocal of a fraction by moving the top to the bottom and the bottom to the top.) Here's why it works.

First notice what happens if you multiply a fraction by its reciprocal.

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ba} = 1$$

Now let's see what happens if a fraction is divided by a fraction. We'll use our observation that $\frac{a}{b} \cdot \frac{b}{a} = 1$.

$$\frac{c/d}{b/a} = 1 \cdot \frac{c/d}{b/a} = \frac{a/b}{a/b} \cdot \frac{c/d}{b/a} = \frac{(a/b)(c/d)}{(a/b)(b/a)} = \frac{(a/b)(c/d)}{1} = (a/b)(c/d)$$

We've done exactly what I said we would do. We multiplied the top fraction by the reciprocal of the bottom fraction.

Now we see that dividing four by a half is the same as multiplying by two.

$$4 \div \frac{1}{2} = \frac{4/1}{1/2} = \frac{4}{1} \cdot \frac{2}{1} = 4 \cdot 2 = 8$$

One more thing about fractions, we always try to write them in “lowest terms.” This means that we write the fraction so that the numerator and denominator have no common factors. Here are a couple of examples.

The fraction $\frac{2}{4}$ is not in lowest terms. Both the numerator and the denominator have a factor of 2 in them. We factor it out and “cancel” as follows.

$$\frac{2}{4} = \frac{2 \cdot 1}{2 \cdot 2} = \frac{2}{2} \cdot \frac{1}{2} = 1 \cdot \frac{1}{2} = \frac{1}{2}$$

This one is a little more difficult.

$$\frac{x^2 + 4x + 4}{x^2 - x - 6}$$

We begin by noticing that the numerator and denominator have a common factor of $x + 2$. Now we may proceed as in the previous example.

$$\frac{x^2 + 4x + 4}{x^2 - x - 6} = \frac{(x + 2)(x + 2)}{(x + 2)(x - 3)} = \frac{x + 2}{x + 2} \cdot \frac{x + 2}{x - 3} = \frac{x + 2}{x - 3}$$

Algebra Hazard # 3: Magic Cancellation in Fractions.

Note that $\frac{a+2b}{3b} \neq \frac{a+2}{3}$ and $\frac{6x+1}{6} \neq x + 1$.

3.1 Problems

Use the rules of basic algebra to write the following expressions as fractions in lowest terms.

12. $\frac{a}{b} - \frac{b}{a}$ (answer : $\frac{a^2-b^2}{ab}$)

13. $\frac{3}{x-2} + \frac{1}{2-x}$ (answer : $\frac{2}{x-2}$)

14. $\frac{x^2+5x+6}{x+3}$

15. $\frac{1}{1+\frac{1}{x-1}}$ (answer : $\frac{x-1}{x}$)

16. $\frac{x}{xy^2} + \frac{y}{x^2y}$

17. $\frac{4a}{b} + \frac{b}{4a}$

Skim back through this worksheet and circle anything you came across that was particularly useful and any ideas that seem particularly important to you, especially anything that was unfamiliar. We will use this worksheet as a reference during the rest of the course. Being consistent with algebra will make everything else go a lot smoother.