

# DOUBLE BUBBLES IN $S^3$ AND $H^3$

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## ABSTRACT

We prove the double bubble conjecture in the three-sphere  $\mathbf{S}^3$  and hyperbolic three-space  $\mathbf{H}^3$  in the cases where we can apply Hutchings theory:

- in  $\mathbf{S}^3$ , each enclosed volume and the complement occupy at least 10% of the volume of  $\mathbf{S}^3$ ;
- in  $\mathbf{H}^3$ , the smaller volume is at least 85% that of the larger.

A balancing argument and asymptotic analysis reduce the problem in  $\mathbf{S}^3$  and  $\mathbf{H}^3$  to some computer checking. The computer analysis has been designed and fully implemented for both spaces.

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## 1. INTRODUCTION

1.1. **The double bubble conjecture in  $\mathbf{S}^3$ ,  $\mathbf{H}^3$ .** In March of 2002, Hutchings, Morgan, Ritoré, and Ros [HMRR] proved that the area-minimizing way to enclose and separate two given volumes in  $\mathbf{R}^3$  is by a standard double bubble, defined as three spherical caps meeting in threes at 120 degree angles. In 2003, Cotton and Freeman extended these methods to  $\mathbf{S}^3$  and  $\mathbf{H}^3$ , proving that the standard double bubble is the most efficient way to enclose and separate two equal volumes in these historically important non-Euclidean spaces, with the added condition that in  $\mathbf{S}^3$

the exterior of the double bubble takes up at least 10% of the volume of  $S^3$ . Our Main Theorem in  $S^3$  (Theorem 6.3) improves upon Cotton and Freeman's results when  $v, w$  are sufficiently large, showing that the standard double bubble is the least-area way to enclose and separate two regions of prescribed unequal volumes, whenever each region and the exterior contains at least ten percent of the total volume. In  $H^3$ , our Main Theorem 6.4, extends Cotton and Freeman's results, showing double bubbles are standard whenever the smallest region has volume at least 0.85 that of the larger region. Computer plots 1 and 2 show that this is approximately the largest range on which current methods of proof using the positivity of the Hutchings function can work. Indeed, the main focus of this paper is rigorously proving that the Hutchings function is positive on most of the regions where it appears positive in these plots. The plots suggest that  $F(v, w)$  is always positive when  $v$  is the volume of the larger region. This fact is shown in Remark 2.21, answering the open question of Cotton and Freeman [CF, Open Question 1.1].

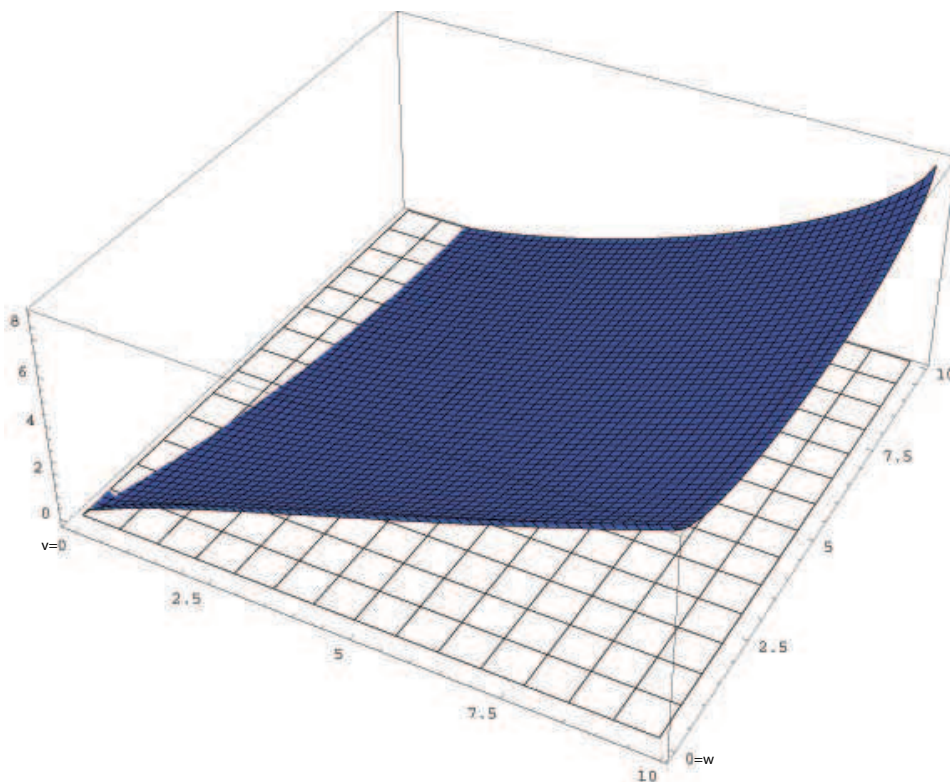


FIGURE 1. Plot of the Hutchings function  $F(v, w)$  where it is positive in  $S^3$ .

**The proof.** The main difficulty is showing that both regions of a minimizing double bubble and the complement all have at most one component each. Given such connectedness, Cotton and Freeman [CF] extend the instability argument of [HMRR] to show that a minimizing double bubble must be standard. By work of Hutchings

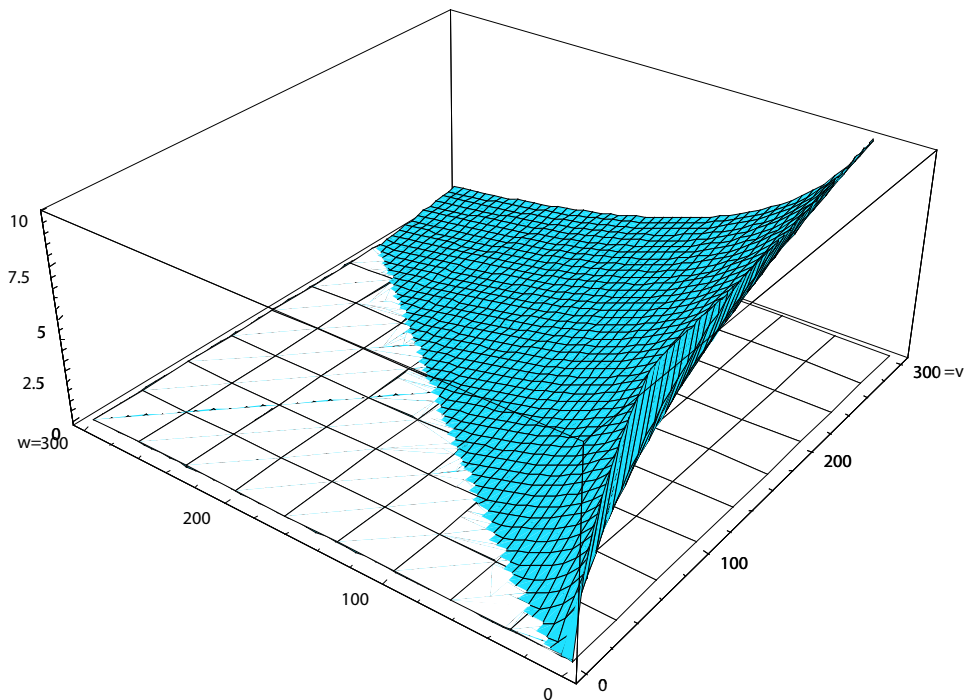


FIGURE 2. Plot of the Hutchings function  $F(v, w)$  where it is positive in  $\mathbf{H}^3$ .

[H], the first region in a minimizer of volumes  $v, w$  is known to be connected if a certain “Hutchings function”  $F(v, w)$ , determined implicitly by integral formulas for volumes of spheres and standard double bubbles (see Definition 2.5), is positive. In their paper, Cotton and Freeman use asymptotic analysis and intensive computation to verify this result along most of the line  $v = w$  in  $\mathbf{S}^3$  and all of the line  $v = w$  in  $\mathbf{H}^3$ , thus proving that the standard double bubble is standard for equal volumes. For the two-dimensional domain of volumes  $(v, w)$  in our unequal volume conjecture, the proof that  $F(v, w) > 0$  is considerably harder. Sections 3 and 4 contain the requisite asymptotic analysis for the non-compact space  $\mathbf{H}^3$ . For small volumes, we use a Euclidean approximation (Section 3), and for large volumes we obtain the interesting result (Proposition 4.12) that

$$\lim_{w \rightarrow \infty} F(\psi w, w) = 2\pi \ln \frac{4(\psi + 1)}{e^2},$$

which is positive if and only if  $\psi > \lambda = e^2/4 - 1 < .85$ . Along the line  $v = \lambda w$ ,  $F$  is *decreasing* in  $w$ , when  $w \geq 300$ . Moreover, once  $w > 150$ ,  $\partial F/\partial v > 0$ . This asymptotic analysis reduces the problem in  $\mathbf{H}^3$  to a bounded domain. We are able to use a computer to show that the Hutchings function is positive on this bounded domain. In  $\mathbf{S}^3$ , which is a space of finite volume, our conjecture requires each region to contain at least 10% of the total volume of  $\mathbf{S}^3$ , and we are again examining a bounded domain. Next we use a new balancing argument (Proposition

2.18), which says that if  $F(\frac{v+w}{2}, \frac{v+w}{2})$  is positive, so is  $F(v, w)$  for  $v > 2w$ . This argument, together with symmetry (used only for  $\mathbf{S}^3$ ) and a balancing argument due to Hutchings (Lemma 2.19), reduces the size of the domain to check. We then break up the domain into small rectangles (and triangles). Finally, we decompose the Hutchings function as the difference of a concave function and an increasing function, so that it suffices to consider the values on (or beyond) the corners of the rectangles (see Lemma 5.1). We find values of the parameters to locate such points and verify that  $F(v, w) > 0$ , allowing a safe margin of error. If necessary, we subdivide the rectangles and repeat the process.

Though using a computer introduces error, we bound this computation error and account for it.

**1.2. Acknowledgments.** The authors would like to thank Colin Adams for posing this problem and for his insightful questions and helpful conversations. We would also like to thank David Futer, Matt Kudzin, and Pat McDonald for helpful conversations. Finally, we would like to thank Frank Morgan for his direction and advice.

This research is the joint work of undergraduate members of the SMALL Geometry Group from '01, '02 and '03, and was completed by Neil Hoffman (Geometry Group '03) in his undergraduate thesis work [Ho]. We thank the National Science Foundation, Williams College, and the organizers of the SMALL REU for helping to make this research possible.

## 2. THE HUTCHINGS FUNCTION IN $\mathbf{S}^3$ AND $\mathbf{H}^3$

**2.1. Notation and definitions.** The pair of positive volumes enclosed and separated by a double bubble in  $\mathbf{S}^3$  must of course satisfy  $v + w < |\mathbf{S}^3|$ , where  $|\mathbf{S}^3|$  denotes the volume of  $\mathbf{S}^3$ . In  $\mathbf{S}^3$ , we put  $u = |\mathbf{S}^3| - v - w$ . The region of volume  $u$  is the complement of the double bubble enclosing volumes  $v$  and  $w$ . We use  $\bar{v}$ ,  $\bar{w}$ , and  $\bar{u}$  to denote  $v/|\mathbf{S}^3|$ ,  $w/|\mathbf{S}^3|$ , and  $u/|\mathbf{S}^3|$  respectively.

**Definition 2.1.** The *least area function* in  $\mathbf{S}^3$  or  $\mathbf{H}^3$  is  $A$ , which maps a volume or a pair of volumes to the least amount of area needed to enclose and separate those volumes in the given space.

**Remark 2.2.** It is well known that the minimal enclosures of single volumes in  $\mathbf{S}^3$  and  $\mathbf{H}^3$  are spheres ([Schmidt], [Morgan, p. 127]); thus  $A_{\mathbf{S}^3}$  takes a single volume to the surface area of a sphere that contains that volume. (Elsewhere, the least area function in *one* argument is known as the *isoperimetric profile*.)

**Remark 2.3.** In a smooth Riemannian manifold  $M$  with compact quotient  $M/\Gamma$  by the isometry group  $\Gamma$ , for any two volumes  $v$  and  $w$  (where  $v + w \leq \text{vol}(M)$  if  $M$  is compact) there exists a least-area enclosure of the two volumes. This enclosure consists of smooth constant-mean-curvature hypersurfaces, except possibly for a set of measure zero ([Morgan, Theorem 13.4, Remark before Proposition 13.8], [CF, Proposition 2.3]).

**Remark 2.4.** The minimal enclosures for any two prescribed volumes in  $\mathbf{S}^3$  and  $\mathbf{H}^3$  are symmetric about some line ([H, Lemma 2.9, Remark 3.8]).

**Definition 2.5.** The *Hutchings function* in  $\mathbf{S}^3$  is  $F_{\mathbf{S}^3} : \{(v, w) : \bar{v} + \bar{w} < 1\} \rightarrow \mathbf{R}$ , where

$$(1) \quad F_{\mathbf{S}^3}(v, w) = A_{\mathbf{S}^3}\left(\frac{v}{2}\right) + A_{\mathbf{S}^3}(w) + A_{\mathbf{S}^3}(v + w) - A_{\mathbf{S}^3}(v, w).$$

The definition of  $F_{\mathbf{H}^3}$  is similar.

**2.2. The Hutchings component bound.** In this section, we recall for the reader the basic property of the Hutchings function, namely that it can be used to limit the number of components in an area-minimizing double bubble.

We suppress the subscripts on  $F$  and  $A$ , since the results stated here apply to both  $\mathbf{S}^3$  and  $\mathbf{H}^3$ .

**Proposition 2.6.** *If  $F(v, w)$  is positive, then in an area-minimizing double bubble enclosing and separating volumes  $v$  and  $w$ , the region  $V$  is connected.*

The reader is referred to [CF, Proposition 4.8] for the proof. The argument given there is essentially the same as the argument in  $\mathbf{R}^n$  ([Morgan, p. 147], [H, Theorem 4.2]).

**Remark 2.7.** We can get a simple lower bound on  $F(v, w)$  by using in place of  $A(v, w)$  the area of a standard double bubble, which will of course be greater than or equal to the area of the least-area double bubble. In the future, when we refer to  $F(v, w)$ , we will be referring to this bound. Similarly, we will (from now on) denote the area of a standard double bubble enclosing the volumes  $v$  and  $w$  as  $A(v, w)$ . Also, it is well known that  $A(v)$  is the area of the sphere containing  $v$ , hence we can make this substitution rigorously.

**2.3. Area and volume formulas.** This section recalls the formulas for area and volume of area-minimizing single bubbles (Remark 2.8). We begin with a statement of the existence and uniqueness of a standard double bubble enclosing given volumes (Proposition 2.10), followed by Propositions 2.11, 2.12, 2.13, and 2.14 which set up basic formulas needed to compute area and volume for standard double bubbles. Single bubbles. Here we present the standard results on volume and surface area of spheres in  $\mathbf{S}^3$  and  $\mathbf{H}^3$ .

**Remark 2.8.** The formulas for surface area of a sphere of radius  $r$  in  $\mathbf{S}^3$  and  $\mathbf{H}^3$  are

$$(2) \quad A_{\mathbf{S}^3} = 4\pi \sin^2 r$$

$$(3) \quad A_{\mathbf{H}^3} = 4\pi \sinh^2 r.$$

The volume formulas for a ball of radius  $r$  in  $\mathbf{S}^3$  and  $\mathbf{H}^3$  are

$$(4) \quad V_{\mathbf{S}^3} = \pi(2r - \sin 2r)$$

$$(5) \quad V_{\mathbf{H}^3} = \pi(\sinh 2r - 2r).$$

**Lemma 2.9.** *The mean curvatures,  $\frac{dA}{dV}$ , of spheres of radius  $r$  in  $\mathbf{S}^3$  and  $\mathbf{H}^3$  are  $2 \cot r$  and  $2 \coth r$  respectively.*

*Proof.* Volume and area are related by  $\frac{dV}{dr} = A$ . Thus, in  $S^3$ , by Remark 2.8,

$$\begin{aligned}
 (6) \quad \frac{dA}{dV} &= \frac{dA/dr}{dV/dr} \\
 (7) &= A'(r)/A(r) \\
 (8) &= \frac{8\pi \sin r \cos r}{4\pi \sin^2 r} \\
 (9) &= 2 \cot r.
 \end{aligned}$$

The proof in  $H^3$  is analogous.  $\square$

Standard double bubbles. Cotton and Freeman [CF, Proposition 2.3] have shown that a standard double bubble in  $S^3$  and  $H^3$  enclosing prescribed volumes  $v$  and  $w$  exists and is unique up to isometry.

**Lemma 2.10** (Standard double bubble is unique). *Given volumes  $v$  and  $w$ :*

- *Up to isometries of the space, there is a unique double bubble in  $S^3$  consisting of three spherical caps meeting at  $120^\circ$  that encloses and separates a region of volume  $v$  and a region of volume  $w$  (whenever  $v + w < 1$ ).*
- *Up to isometries of the space, there is a unique double bubble in  $H^3$  consisting of two spheres (the outer caps) and a sphere, hypersphere, horosphere or geodesic plane (the inner cap) meeting at  $120^\circ$  that encloses and separates a region of volume  $v$  and a region of volume  $w$ .*

Since each standard double bubble in  $S^3$  and  $H^3$  is symmetric about a geodesic line, it is convenient to compute area and volume of a standard double bubble by considering the revolution of some generating curve consisting of three circular arcs meeting at  $120^\circ$ . Figure 3 represents an arbitrary generating curve labeled with relevant measurements.

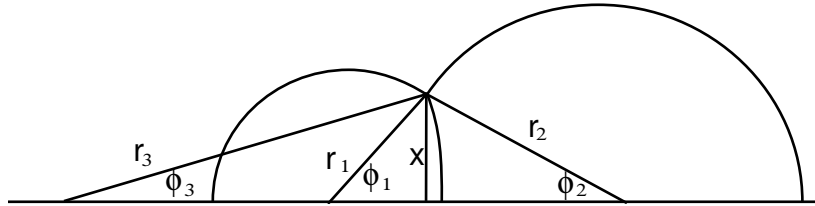


FIGURE 3. Some important geometrical features of the generating curve for a standard double bubble.

**Proposition 2.11** (Area of Spherical Cap in  $S^3$ ). *In  $S^3$  the area of a spherical cap subtended by an angle  $2\phi_0$  is*

$$(10) \quad \int_0^{2\pi} \int_0^{\phi_0} \sin^2 r \sin \phi \, d\phi \, d\theta = 2\pi \sin^2 r (1 - \cos \phi_0).$$

**Proposition 2.12** (Volume of Spherical Cap in  $S^3$ ). *The volume of a spherical cap that is subtended by an angle  $2\phi_0$  is*

$$(11) \quad -\pi(\tan^{-1}(\cos \phi_0 \tan r) - r \cos \phi_0 + (-1 + \cos \phi_0)(r - \cos r \sin r)).$$

### Hyperbolic Space.

**Proposition 2.13** (Volume of Spherical Cap in  $\mathbf{H}^3$ ). *The volume of a spherical cap in a sphere of radius  $r$  subtended by an angle  $2\phi_0$  is*

$$(12) \quad \pi(-r + \tanh^{-1}(\cos \phi_0 \tanh r) - \cos \phi_0 \cosh r \sinh r + \cosh r \sinh r)).$$

**Proposition 2.14** (Area of Spherical Cap in  $\mathbf{H}^3$ ). *In  $\mathbf{H}^3$  the area of a spherical cap in a sphere of radius  $r$  subtended by an angle  $2\phi_0$  is*

$$(13) \quad \int_0^{2\pi} \int_0^{\phi_0} \sinh^2 r \sin \phi d\phi d\theta = 2\pi \sinh^2 r (1 - \cos \phi_0).$$

Now we can compute the volume and areas of standard double bubbles.

For each bubble component, we compute the volume and area of the spherical cap which separates it from the exterior. To find the area of the double bubble, we simply add the area of the three spherical caps.

To compute the volume of the bubble components we add the volume contained by the separating cap to that contained by the smaller of the two outer cap to obtain the volume of the smaller region, and subtract the volume contained by the separating cap from that contained by the larger of the two outer cap to obtain the volume of the larger region. (Of course, when the two outer caps are the same size, the separating cap contains no volume so there is nothing to add or subtract.)

Examining Figure 3, we observe that the angle  $\phi_1$  has a different orientation (relative to  $r_2$  and  $r_3$ ) depending on the relative size of the two bubbles. When  $\phi_1$  is  $90^\circ$ ,  $r_2$  and  $r_3$  are equal causing the volumes enclosed by both regions of the double bubble to be equal. Tracing back through the equations, we notice that this happens when

$$(14) \quad \frac{\tan(r_2) - \tan(r_1)}{\tan(r_2) + \tan(r_1)} = 1/3 \quad (\text{in } \mathbf{S}^3)$$

and

$$(15) \quad \frac{k_1 - k_2}{k_1 + k_2} = 1/3 \quad (\text{in } \mathbf{H}^3).$$

We make adjustments to  $\phi_1$  accordingly, and call the adjusted angle  $\hat{\phi}_1$ . Using this angle in the formulas for area and volumes of spherical caps yields the desired result, i.e., if  $\hat{\phi}_1 > \frac{\pi}{2}$ , the formulas return values corresponding to a cap smaller than a hemisphere, otherwise the values returned describe a cap that is larger than a hemisphere.

#### 2.4. Properties of Hutchings function.

**Proposition 2.15.**  *$A(v)$ ,  $A(v,w)$  in  $\mathbf{S}^n$  and  $\mathbf{H}^n$   $n \geq 2$  are strictly concave functions.*

*Proof.* See [H, Remark 3.8 and Theorem 3.9]. □

**Proposition 2.16.**  *$A(v,w)$  in  $\mathbf{S}^3$  is strictly increasing in  $v,w$  on the closed triangular domain defined by  $(0,0), (\frac{|\mathbf{S}^3|}{3}, \frac{|\mathbf{S}^3|}{3}), (0, \frac{|\mathbf{S}^3|}{2})$ . Furthermore,  $A(v,w)$  is also increasing on the line segment with end points  $(0, \frac{|\mathbf{S}^3|}{2}), (\frac{|\mathbf{S}^3|}{3}, \frac{|\mathbf{S}^3|}{3})$ .*

*Proof.* Since  $A(v, w)$  is concave (Proposition 2.15) and symmetric in both arguments,  $A(v, w)$  in  $\mathbf{S}^3$  attains its maximum at  $(\frac{|\mathbf{S}^3|}{3}, \frac{|\mathbf{S}^3|}{3})$ . Since  $A(v, w)$  is also concave along lines through  $(\frac{|\mathbf{S}^3|}{3}, \frac{|\mathbf{S}^3|}{3})$  it is increasing on the triangular domain and the line segment with end points  $(0, \frac{|\mathbf{S}^3|}{2}), (\frac{|\mathbf{S}^3|}{3}, \frac{|\mathbf{S}^3|}{3})$ .  $\square$

**Proposition 2.17.** *For all  $v, w$ ,  $A_{\mathbf{H}^3}(v, w)$  is strictly increasing in each variable.*

*Proof.*  $A_{\mathbf{H}^3}(v, w)$  is strictly concave along any line, as shown in Lemma 2.15: in particular, along any line  $v = v_0$  or  $w = w_0$ . Because it is positive for all  $v, w > 0$ , it follows that it can never be non-increasing along any of these lines.  $\square$

**2.5. Balancing.** The argument given here shows, for example, that if the regions in equal volume double bubbles are connected, then the largest region of any double bubble in  $\mathbf{S}^n$ ,  $\mathbf{H}^n$ , or  $\mathbf{R}^n$ , for  $n \geq 3$ , must be connected. The result can be stated in even more general terms:

**Proposition 2.18 (Balancing).** *In  $\mathbf{R}^n$ ,  $\mathbf{H}^n$ , or  $\mathbf{S}^n$ , for  $n \geq 3$ , if  $w < v < 2w$  and  $F(\frac{v+w}{2}, \frac{v+w}{2}) > 0$ , then  $F(v, w) > 0$ .*

*Proof.* Consider an area pair  $v > w$  and suppose that  $F(\frac{v+w}{2}, \frac{v+w}{2})$  is positive, i.e., that

$$(16) \quad 2A\left(\frac{v+w}{4}\right) + A\left(\frac{v+w}{2}\right) + A(v+w) - 2A\left(\frac{v+w}{2}, \frac{v+w}{2}\right) > 0.$$

Concavity and the symmetry of  $A$  implies that for any  $v$  not equal to  $w$ ,

$$A\left(\frac{v+w}{2}, \frac{v+w}{2}\right) > A(v, w).$$

But then we have

$$F(v, w) > 2A\left(\frac{v}{2}\right) + A(w) + A(v+w) - 2A\left(\frac{v+w}{2}, \frac{v+w}{2}\right).$$

So it suffices to show that

$$\begin{aligned} & 2A\left(\frac{v}{2}\right) + A(w) + A(v+w) - 2A\left(\frac{v+w}{2}, \frac{v+w}{2}\right) \\ & > 2A\left(\frac{v+w}{4}\right) + A\left(\frac{v+w}{2}\right) + A(v+w) - 2A\left(\frac{v+w}{2}, \frac{v+w}{2}\right) \end{aligned}$$

i.e. that

$$(17) \quad 2A\left(\frac{v}{2}\right) + A(w) > 2A\left(\frac{v+w}{4}\right) + A\left(\frac{v+w}{2}\right).$$

Consider three spheres, two enclosing volume  $x/2 < y$  and one enclosing volume  $y$ . Initially, increasing  $x$  and decreasing  $y$  while keeping  $x+y$  constant will increase the total surface area (we know this because  $2A(x/2) + A(y(x))$  is a concave function). Indeed, area will increase until  $x/2 = y$ , that is, until all three spheres have equal volume and any further transition would be *unbalancing*. In particular, if  $w < v < 2w$  and we put  $x = y = (v+w)/2$ , then increasing  $x$  until it equals  $v$  while decreasing  $y$  until it equals  $w$  will increase the total surface area of our three spheres. This shows (Equation 17).

Hence  $F(v, w) > 0$  for  $w < v < 2w$ .

□

**Lemma 2.19** (Hutchings Balancing). *In  $\mathbf{R}^n$ ,  $\mathbf{H}^n$ , or  $\mathbf{S}^n$ , for  $n \geq 3$ , if  $v > 2w$  (or in  $\mathbf{S}^3$  if  $v > 2u$ ), then the region of volume  $v$  is connected.*

*Proof.* See [H, Remark 3.8, Theorem 3.5, Corollary 3.9].

□

**2.6. Permutation.** Permutation utilizes symmetries that exist with the Hutchings function when applied in compact spaces. Though this argument is applied in  $\mathbf{S}^3$ , it can be used in any space with finite volume that has a Hutchings function.

**Proposition 2.20** (Permutation).  *$F(v, w) = F(v, u)$  where  $u$  is the volume of the complement.*

*Proof.* First notice that  $A(v, w) = A(v, u)$  as it is only a matter of labeling  $U$  or  $W$  the exterior. Now,  $A(v+w) = A(u)$  and  $A(v+u) = A(w)$ , since  $U = (V \cup W)^c$ , and similarly,  $W = (V \cup U)^c$ . Thus,  $F(v, w) = 2A(\frac{v}{2}) + A(w) + A(v+w) - 2A(v, w) = 2A(\frac{v}{2}) + A(w) + A(u) - 2A(v, w) = F(v, u)$ , as desired. □

**Remark 2.21.** Balancing (Proposition 2.18) and Permutation (Proposition 2.20) yield a complete answer to a question of Cotton and Freeman [CF, Open Question 1]. In particular, the larger region of an area-minimizing double bubble in  $\mathbf{H}^3$  is always connected. This follows from the positivity of the Hutchings function  $F(v, w)$ . This in turn follows from Balancing (Proposition 2.18) and the positivity of the Hutchings Function  $F(v, v)$  for equal volumes [CF, Proposition 5.11, 5.14, 5.19]. In  $\mathbf{S}^3$ , it suffices to check that  $F(v, v) > 0$  for  $\bar{v} \leq .5$ . Cotton and Freeman [CF, Proposition 5.5, 5.8] show this to be true for  $\bar{v} \leq .45$ . For  $\bar{v} \in (.45, .5]$ , by Permutation (Proposition 2.20) it suffices to show  $F(v, 1-2v) > 0$ , which follows from Balancing (Proposition 2.18).

### 3. POSITIVITY OF THE HUTCHINGS FUNCTION FOR SMALL VOLUMES IN $\mathbf{H}^3$

#### 3.1. Properties of the Hutchings function in $\mathbf{H}^3$ .

**Theorem 3.1** (Double Bubble Theorem in  $\mathbf{R}^3$  [HMRR]). *The unique least-area enclosure of prescribed volumes  $v, w$  in  $\mathbf{R}^3$  is a standard double bubble.*

**Remark 3.2.** For small prescribed volumes in  $\mathbf{H}^3$ , the space around any standard double bubble is nearly flat and looks like a portion of  $\mathbf{R}^3$ . By estimating the amount of distortion carefully, we can use information about the Hutchings function in  $\mathbf{R}^3$  to prove that the Hutchings function in  $\mathbf{H}^3$  is positive for certain prescribed volumes near 0. In the following proof,  $\lambda$  is a constant which captures the amount of distortion, and  $r$  represents the radius of a small ball in  $\mathbf{H}^3$  which is only slightly affected by the curvature of  $\mathbf{H}^3$ .

**Lemma 3.3.** *For any pair of prescribed volumes less than  $(\frac{2\pi}{3} + \frac{\pi\sqrt{3}}{2})(\frac{2}{2+\sqrt{3}}r)^3$ , some minimizing double bubble in  $\mathbf{R}^3$  enclosing those volumes fits inside a ball of radius  $r$ .*

*Proof.* Without loss of generality assume that  $v \leq w$  and fix  $w$ . The exterior spherical cap enclosing the region of volume  $w$  has a radius,  $r_w$ . Consider standard double bubbles where the radius of this spherical cap is fixed at  $r_w$ . In this case,

the equal volumes double bubble has the greatest diameter. However, the region enclosed by the spherical cap with radius  $r_w$  has volume less than  $w$ . By increasing the radius until this region has volume  $w$ , the diameter increases. Thus for any non-equal volume double bubble enclosing volumes  $v$  and  $w$ , where  $v \leq w$ , there will be an equal volume double bubble enclosing  $v'$  and  $w$  with a bigger diameter.

If  $r_w$  is the radius of the spherical cap that forms the exterior of  $W$ , then the diameter of the equal volume double bubble is  $2r_w + r_w\sqrt{3}$ . So, this double bubble will fit inside a ball of radius  $r > r_w \frac{(2+\sqrt{3})}{2}$ .

We also know that in  $\mathbf{R}^3$  a spherical cap of radius  $r_0$  subtended by an angle  $2\phi$  has volume:

$$(18) \quad V(r) = \frac{2\pi}{3}(r_0)^3 + \pi r_0^3 \cos\phi.$$

Since  $\phi \in [0, \frac{\pi}{6}]$ ,

$$(19) \quad w \leq (r_w)^3 \left( \frac{2\pi}{3} + \frac{\pi\sqrt{3}}{2} \right).$$

Substituting  $\frac{2}{2+\sqrt{3}}r$  for  $r_w$  in the above equation gives the desired result.  $\square$

**Lemma 3.4.** *If  $.84w \leq v \leq w$ , then*

$$2A_{\mathbb{R}^3}\left(\frac{v}{2}\right) + A_{\mathbb{R}^3}(w) + A_{\mathbb{R}^3}(v+w) > (2.02676)A_{\mathbb{R}^3}(v, w)$$

*holds.*

*Proof.* (Modification of [HMRR, Proposition 6.2]) By rescaling, we may assume that  $v = 1 - w$  and that  $w \in [\frac{1}{2}, \frac{1}{1.84}]$ . Because  $A_{\mathbb{R}^3}(v, w)$  is concave [H, Theorem 3.2], we have

$$A_{\mathbb{R}^3}(1-w, w) \leq A_{\mathbb{R}^3}\left(\frac{1}{2}, \frac{1}{2}\right) = (2^{-\frac{4}{3}} \cdot 3)A_{\mathbb{R}^3}(1).$$

Hence, it suffices to prove that

$$2A_{\mathbb{R}^3}\left(\frac{1-w}{2}\right) + A_{\mathbb{R}^3}(w) + A_{\mathbb{R}^3} \geq 2.02676(2^{-\frac{4}{3}} \cdot 3)A_{\mathbb{R}^3}(1),$$

or equivalently (dividing by  $A_{\mathbb{R}^3}(1)$  on both sides) that

$$2^{\frac{1}{3}}(1-w)^{\frac{2}{3}} + w^{\frac{2}{3}} + 1 \geq 2.02676(2^{-\frac{4}{3}} \cdot 3).$$

At  $w = \frac{1}{2}$  and  $w = \frac{1}{1.84}$ , the left hand side of the above inequality is bigger than 2.4236 and 2.412965, respectively. The right hand side of the above inequality is less than 2.412966. Hence, the inequality holds for  $w = \frac{1}{2}$  and  $w = \frac{1}{1.84}$ ; and because the left hand side is concave in  $w$ , the inequality holds for all  $w \in [\frac{1}{2}, \frac{1}{1.84}]$ , as desired.  $\square$

**Proposition 3.5.** *If  $0 < v, w < .002743$  and  $.84w \leq v \leq w$ , then*

$$(20) \quad 2A_{\mathbf{H}^3}\left(\frac{v}{2}\right) + A_{\mathbf{H}^3}(w) + A_{\mathbf{H}^3}(v+w) - 2A_{\mathbf{H}^3}(v, w) > 0.$$

*Proof.* By Lemma 3.3, we can fit a double bubble in  $\mathbf{R}^3$  with prescribed volumes  $v, w < .002743$  in a ball  $B$  of radius  $r < .1547$ .

For ease of notation let  $\lambda = \frac{\sinh r}{r} < 1.003994$ .

Notice that if this ball can enclose a standard double bubble of volumes  $(\lambda^{-2}v, \lambda^{-2}w)$  then it can enclose spheres of volumes:

$\{\lambda^{-2}\frac{v}{2}, \lambda^{-2}w, \lambda^{-2}(v+w)\}$ . We may estimate the surface area of any sphere in  $\mathbf{R}^3$  which fits inside  $B$ . Suppose that a sphere enclosing volume  $\lambda^{-2}\alpha$  fits inside  $B$ ; in particular, this holds for  $\alpha \in \{\frac{v}{2}, w, (v+w)\}$ . The single bubble in  $\mathbf{H}^3$  encloses a volume  $\alpha' \leq \alpha$  and has surface area at least  $A_{\mathbf{R}^3}(\lambda^{-2}\alpha)$ . Thus,

$$(21) \quad A_{\mathbf{H}^3}(\alpha) \geq A_{\mathbf{R}^3}(\lambda^{-2}\alpha).$$

Therefore,

$$\begin{aligned} & 2A_{\mathbf{H}^3}\left(\frac{v}{2}\right) + A_{\mathbf{H}^3}(w) + A_{\mathbf{H}^3}(v+w) \\ & \geq 2A_{\mathbf{R}^3}(\lambda^{-2}v/2) + A_{\mathbf{R}^3}(\lambda^{-2}w) + A_{\mathbf{R}^3}(\lambda^{-2}(v+w)). \end{aligned}$$

By Lemma 3.4,

$$\begin{aligned} & 2A_{\mathbf{R}^3}\left(\frac{\lambda^{-2}v}{2}\right) + A_{\mathbf{R}^3}(\lambda^{-2}w) + A_{\mathbf{R}^3}(\lambda^{-2}(v+w)) \\ & \geq (2.02676)A_{\mathbf{R}^3}(\lambda^{-2}v, \lambda^{-2}w). \end{aligned}$$

Scaling of double bubbles in  $\mathbf{R}^3$  tells us

$$(22) \quad A_{\mathbf{R}^3}(\lambda^{-2}v, \lambda^{-2}w) = (\lambda)^{-4/3}A_{\mathbf{R}^3}(v, w).$$

Because  $B$  contains some minimizing double bubble  $\Sigma_1$  in  $\mathbf{R}^3$  enclosing volumes  $v, w$ , we can estimate the surface area of the image of  $\Sigma_1$  in  $\mathbf{H}^3$ . This double bubble  $\Sigma_2$  encloses volumes  $v_2 \geq v$  and  $w_2 \geq w$ , with surface area less than or equal to  $A_{\mathbf{R}^3}(v_2, w_2)\lambda^2$ . Therefore,

$$(23) \quad A_{\mathbf{R}^3}(v, w) \geq \lambda^{-2}A_{\mathbf{H}^3}(v_2, w_2).$$

By Proposition 2.17,  $A_{\mathbf{H}^3}(x, y)$  is increasing in each variable, implying that

$$(24) \quad A_{\mathbf{H}^3}(v_2, w_2) \geq A_{\mathbf{H}^3}(v, w).$$

Combining inequalities (22), (23), and (24) with equality (22) gives

$$(25) \quad 2A_{\mathbf{H}^3}\left(\frac{v}{2}\right) + A_{\mathbf{H}^3}(w) + A_{\mathbf{H}^3}(v+w) \geq (2.0676)(\lambda)^{-10/3}A_{\mathbf{H}^3}(v, w).$$

Note that  $(2.02676)\lambda^{-10/3} > 2$ , so

$$(26) \quad 2A_{\mathbf{H}^3}\left(\frac{v}{2}\right) + A_{\mathbf{H}^3}(w) + A_{\mathbf{H}^3}(v+w) > 2A_{\mathbf{H}^3}(v, w)$$

as desired. □

**Remark 3.6** (Notes on  $\mathbf{S}^3$ ). The above argument works for  $\mathbf{S}^3$  as well, although we do not need it because we are just concerned with volumes greater than 10% of the total volume of  $\mathbf{S}^3$ . The  $\mathbf{S}^3$  proof uses a different distortion factor,  $\lambda = \frac{\sin r}{r}$ . Otherwise the proof follows with very little adjustment. Using Lemma 3.4, the Hutchings Function is positive for small volumes in  $\mathbf{S}^3$  namely  $v, w < 0.002738$  and  $.84w \leq v \leq w$ .

#### 4. POSITIVITY OF THE HUTCHINGS FUNCTION FOR LARGE VOLUMES IN $\mathbf{H}^3$

**4.1. Introduction.** Theorem 4.23 shows that for sufficiently large  $v$  and  $w$ , the Hutchings function in  $\mathbf{H}^3$  is positive whenever  $\psi = v/w \geq \lambda$  and  $w \geq 300$ . Throughout this section we set  $\lambda = e^2/4 - 1$ .

The proof of the main result works with a modification of the Hutchings function which is less than or equal to the original Hutchings function. The proof of Theorem 4.23 uses three ancillary results:

Proposition 4.12 . *For a fixed ratio  $v/w$ , the limit of the modified Hutchings function as  $w \rightarrow \infty$  is nonnegative if and only if  $\psi \geq \lambda < .85$ .*

Proposition 4.20 . *The modified Hutchings function strictly decreases if one travels outward along the line  $v = \lambda w$  as long as  $w > 300$ .*

Lemma 4.22 . *The partial derivative with respect to  $v$  of the modified Hutchings function is positive if  $w \geq 150$ ,  $v \geq \lambda w$ .*

Together, these results show that the modified Hutchings function (and hence the Hutchings function) is positive for  $\psi \geq \lambda$  and  $w > 300$ . In order to prove Proposition 4.12, we must describe the limits of various quantities for standard double bubbles enclosing volumes  $v, w$  as  $v, w$  grow large. Proving Proposition 4.20 and Lemma 4.22 require a closer examination of how fast these quantities converge to their limits. So, the results about the limits appear alongside various inexact numerical estimates.

**4.2. Preliminary material: spheres in  $\mathbf{H}^3$ .** If  $r = r(v)$  is the radius of a sphere with volume  $v$ , then

$$(27) \quad 2v/\pi = e^{2r} - e^{-2r} - 4r < e^{2r}$$

so that

$$(28) \quad r > \frac{1}{2} \ln \frac{2v}{\pi}.$$

**Lemma 4.1.** *If  $r = r(v)$  is the radius of a sphere with volume  $v$ , then*

$$(29) \quad \lim_{v \rightarrow \infty} \left( r - \frac{1}{2} \ln \frac{2v}{\pi} \right) = 0.$$

*Proof.* By manipulation of the volume formula,

$$(30) \quad \frac{2v}{\pi} (e^{2(r - \frac{1}{2} \ln \frac{2v}{\pi})} - 1) = e^{-2r} + 4r.$$

Notice for any  $\epsilon > 0$  we can find a sufficiently large  $r$ , such that,

$$(31) \quad e^{-2r} + 4r < 2(\sinh 2r - 2r)(e^{2\epsilon} - 1).$$

Hence,  $\frac{2v}{\pi} (e^{2(r - \frac{1}{2} \ln \frac{2v}{\pi})} - 1)$  is less than  $\frac{2v}{\pi} (e^{2\epsilon} - 1)$  and  $2(r - \frac{1}{2} \ln \frac{2v}{\pi}) < 2\epsilon$ , as needed.  $\square$

**Remark 4.2.** When  $v > 150\lambda > \pi(\sinh(4.4) - 4.4)$ , we have  $r > 2.2$  and Inequality (31) holds for  $\epsilon = 0.06$ . Thus,  $r < \frac{1}{2} \ln \frac{2v}{\pi} + .06$  for such  $v$ . We obtain the following numerical estimate: if  $v > 150\lambda < 127.09$ , then  $r < \frac{1}{2} \ln \frac{2v}{\pi} + .06$ .

**Lemma 4.3.** *As  $v$  approaches infinity,  $A(v)$  tends to  $2v + 2\pi \ln v - 2\pi(1 - \ln(\pi/2))$ .*

*Proof.* Let  $r = r(v)$  denote the radius of a sphere with volume  $v$ . Since

$$(32) \quad r = \frac{1}{2} \ln \left( \frac{2v}{\pi} + e^{-2r} + 4r \right),$$

we see that,

$$(33) \quad A(v) = 2v - 2\pi + 2\pi e^{-2r} + 4\pi r.$$

By Lemma 4.1,

$$(34) \quad \lim_{v \rightarrow \infty} r - \frac{1}{2} \ln \frac{2v}{\pi} = \lim_{v \rightarrow \infty} r - \frac{1}{2} \ln v + \frac{1}{2} \ln \frac{\pi}{2} = 0.$$

Also,  $\lim_{v \rightarrow \infty} e^{-2r} = 0$ . The desired result follows easily.  $\square$

**Lemma 4.4.** *The curvature of a sphere with volume  $v$  and radius  $r$  is*

$$(35) \quad 2 + \frac{2\pi}{v + 2\pi r + \frac{\pi}{2}e^{-2r} - \pi/2}.$$

*Proof.* The curvature equals

$$(36) \quad 2 \coth r = 2 \frac{e^r + e^{-r}}{e^r - e^{-r}} = 2 + \frac{4e^{-r}}{e^r - e^{-r}} = 2 + \frac{4}{e^{2r} - 1}.$$

Also,  $2v/\pi = e^{2r} - e^{-2r} - 4r$ . The desired result follows easily.  $\square$

**Lemma 4.5.**  $A'(x) < 2 + \frac{2\pi}{x + \pi \ln x - 3}$ .

*Proof.* From (28) we have the following bound on the radius  $r$  of a sphere containing volume  $x$ :

$$(37) \quad r > \frac{1}{2} \ln(2x/\pi)$$

$$(38) \quad = \frac{1}{2} \ln x + (1/2) \ln(2/\pi)$$

$$(39) \quad > \frac{1}{2} \ln x - 3/2\pi + 1/4.$$

By Lemma 4.4,

$$(40) \quad A'(x) = 2 + \frac{2\pi}{x + 2\pi r + (\pi/2)e^{-2r} - \pi/2}.$$

Also,  $e^{-2r} > 0$ . Applying these inequalities in the above expression for  $A'(x)$  yields the desired result.  $\square$

**Lemma 4.6.** *If  $x > 150\lambda$  then  $A'(x) > 2 + \frac{2\pi}{x + \pi \ln x - 1.041}$ .*

*Proof.* By Lemma 4.4,

$$(41) \quad A'(v) = 2 + \frac{2\pi}{v + 2\pi r + \frac{\pi}{2}e^{-2r} - \frac{\pi}{2}}.$$

The denominator is a sum of three terms:

$$v, 2\pi r, \frac{\pi}{2}e^{-2r} - \frac{\pi}{2}.$$

Because  $e^{-2r} < 1$ , the third term is less than 0. We now approximate the second term. By Remark 4.2, we have  $r < \frac{1}{2} \ln \frac{2v}{\pi} + 0.06$  because  $v > 150\lambda$ . Hence, the second term satisfies

$$(42) \quad 2\pi r < \pi \ln \frac{2v}{\pi} + 0.12\pi = \pi \ln v + \pi \ln \frac{2}{\pi} + 0.12\pi < \pi \ln v - 1.041.$$

With these approximations, we find

$$(43) \quad A'(v) > 2 + \frac{2\pi}{v + \pi \ln v - 1.041}.$$

□

**Corollary 4.7.** *If  $x > 300\lambda$  then  $A'(x+3) > 2 + \frac{2\pi}{x + \pi \ln x + 2}$ .*

*Proof.* By Lemma 4.6,  $A'(x) > 2 + \frac{2\pi}{x + \pi \ln x - 1.041}$ , if  $x > 150\lambda$ .

Because  $e^{-2r} < 1$ , the third term is less than 0. We now approximate the second term. By Remark 4.2, we have  $r < \frac{1}{2} \ln \frac{2v}{\pi} + 0.06$  because  $v > 150\lambda$ . Hence, the second term satisfies

$$(44) \quad 2\pi r < \pi \ln \frac{2v}{\pi} + 0.12\pi = \pi \ln v + \pi \ln \frac{2}{\pi} + 0.12\pi < \pi \ln v - 1.041.$$

With these approximations, we find

$$(45) \quad A'(v) > 2 + \frac{2\pi}{v + \pi \ln v - 1.041}.$$

For  $x > 300\lambda$ , we have

$$(46) \quad \pi \ln \frac{x+3}{x} - 1.041 < -1.004.$$

Hence,

$$(47) \quad A'(x+3) > 2 + \frac{2\pi}{x+3 + \pi \ln x - 1.004} > 2 + \frac{2\pi}{x + \pi \ln x + 2}.$$

□

#### 4.3. The Hutchings function has a positive limit.

**Lemma 4.8.** *Given a standard double bubble, the radius of the circular interface where the three caps meet is less than  $\cosh^{-1}(2)$ .*

*Proof.* Consider the disc whose boundary is the circular interface. Let  $y$  be the radius of this disk. It meets the bigger of the outer caps of the double bubble at an angle  $\alpha \geq 2\pi/3$ . The radii of this outer cap and the disk come together at an angle of  $\beta \geq \pi/6$  along the circular interface. Let  $2\phi_o$  be the angle that subtends this cap. By the formula for the area of a spherical cap in  $H^3$  (Proposition 2.14) and the hyperbolic laws of cosines and sines [Thurston] and [Stahl],

$$(48) \quad \cos \phi_o = -\cosh y \sin \beta$$

$$(49) \quad \sinh r = \frac{\sinh y}{\sin \phi_o}$$

the surface area of the outer cap is

$$(50) \quad \frac{2\pi \sinh^2 y}{1 - \cosh y \sin \beta}.$$

The surface area of this cap is positive and finite, implying that  $\cosh y < \csc \beta \leq \csc(\pi/6) = 2$ . Because  $t \mapsto \cosh t$  is increasing for  $t > 0$ ,  $y < \cosh^{-1}(2)$ .  $\square$

**Lemma 4.9.** *Given a spherical cap subtended by an angle of  $2\phi$  on a sphere of radius  $r$ , let  $\theta$  be the angle the sphere makes with the disk of radius  $y$  bounding the cap from below. The volume of this cap can be expressed as,*

$$(51) \quad \pi \left( \frac{\sinh y \sin \theta}{\operatorname{sech} y + \cos \theta} - \tanh^{-1} \left( \frac{\sinh y \sin \theta}{\cosh y + \cos \theta} \right) \right).$$

*Proof.* This follows from using the identities:

$$\begin{aligned} \tanh(a+b) &= \frac{\tanh a + \tanh b}{1 + \tanh a \tanh b} \text{ where } (a+b) = (-r + \tanh^{-1}(\tanh r \cos \phi)), \\ 1 - \tanh^2 x &= \operatorname{sech}^2 x \text{ and } \cosh b = \frac{\sin \phi}{\cos \gamma}, \\ \cosh c &= \cosh y \cosh b, \\ \cos \gamma &= \sin \phi \cosh b, \\ \tanh r \cos \gamma &= \tanh y, \quad \cosh y \sin \gamma = \cos \phi, \\ \sinh y &= \sin \phi \sinh r \text{ and } \cos \gamma = \frac{\tanh y}{\tanh r}, \text{ and} \\ \sin \phi &= \frac{\sinh y}{\sinh r}. \end{aligned}$$

$\square$

**Lemma 4.10.** *Given a standard double bubble enclosing volumes  $v, w$ , suppose that  $\theta$  is the angle between the separating cap and the disc with the same boundary, and let  $y$  be the radius of the disc. As  $v, w$  grow large,  $\theta$  approaches 0 and  $y$  approaches  $\cosh^{-1}(2)$ .*

*Proof.* By Lemma 4.8,  $y < \cosh^{-1}(2)$ . The two caps of the region containing the disk meet the given disc of radius  $y$  at angles  $\theta \in [0, \pi/3)$  and  $\alpha = 2\pi/3 - \theta$ . By volume formulas, the volume  $x$  of this region is the sum of

$$(52) \quad \pi \left( \frac{\sinh y \sin \theta}{\operatorname{sech} y + \cos \theta} - \tanh^{-1} \left( \frac{\sinh y \sin \theta}{\cosh y + \cos \theta} \right) \right)$$

and

$$(53) \quad \pi \left( \frac{\sinh y \sin \alpha}{\operatorname{sech} y + \cos \alpha} - \tanh^{-1} \left( \frac{\sinh y \sin \alpha}{\cosh y + \cos \alpha} \right) \right).$$

Notice that  $\cosh y + \cos \theta > 1 - 1 = 0$ , and similarly  $\cosh y + \cos \alpha > 0$ . Hence the “ $\tanh^{-1}$ ” terms are positive, implying that

$$(54) \quad x < \pi \left( \frac{\sinh y \sin \alpha}{\operatorname{sech} y + \cos \alpha} + \frac{\sinh y \sin \theta}{\operatorname{sech} y + \cos \theta} \right).$$

Because  $\operatorname{sech} y > 1/2$  and  $\cos \theta, \cos \alpha > -1/2$ , the denominators of the above expression are positive. Hence,

$$\begin{aligned} x &< \pi \left( \frac{\sinh(\cosh^{-1}(2)) \sin(\pi/2)}{\operatorname{sech} y + \cos \theta} + \frac{\sinh(\cosh^{-1}(2)) \sin(\pi/2)}{\operatorname{sech} y + \cos \alpha} \right) \\ &= \pi \left( \frac{\sqrt{3}}{\operatorname{sech} y + \cos \theta} + \frac{\sqrt{3}}{\operatorname{sech} y + \cos \alpha} \right) \\ &< \frac{2\pi\sqrt{3}}{\operatorname{sech} y + \cos(2\pi/3 - \theta)}. \end{aligned}$$

As  $v, w$  approach infinity,  $x \geq \min\{v, w\}$  must approach infinity as well, implying that the positive quantity  $\operatorname{sech} y + \cos(2\pi/3 - \theta)$  must approach 0 – that is,  $y$  must approach  $\cosh^{-1}(2)$ , and  $\cos(2\pi/3 - \theta)$  must approach  $-\frac{1}{2}$ . Because  $2\pi/3 - \theta \in (\pi/3, 2\pi/3]$ ,  $\theta$  must approach 0. In particular, when  $v, w > x_0 = 300\lambda$ , we have

$$(55) \quad x_0 < \frac{2\pi\sqrt{3}}{\operatorname{sech} y + \cos(2\pi/3 - \theta)} < \frac{2\pi\sqrt{3}}{1/2 + \cos(2\pi/3 - \theta)}$$

or

$$(56) \quad \cos(2\pi/3 - \theta) < \frac{2\pi\sqrt{3}}{x_0} - 1/2 < -.457.$$

For  $\theta \in [0, \pi/3)$ , this is true only if  $\theta < 1/20$ .  $\square$

We have the following numerical estimate: if  $v, w > 300\lambda \approx 254.18$ , then  $\theta < 1/20$ .

**Lemma 4.11.** *As  $v, w$  grow large, the surface area of a standard double bubble enclosing volumes  $v, w$  approaches*

$$(57) \quad A_{H^3}(v + v_\infty) + A_{H^3}(w + v_\infty) - 2a_\infty + c_\infty$$

where  $v_\infty = \pi(3/2 - \ln 2)$ ,  $a_\infty = 3\pi$ , and  $c_\infty = 2\pi$ .

*Proof.* Given a standard double bubble enclosing volumes  $v, w$ , consider the region enclosing volume  $v$ . Its outer cap lies on a sphere; let the volume of this sphere be  $v + v_{v,w}^{(1)}$  and let the surface area of the sphere exceed that of the outer cap by  $a_{v,w}^{(1)}$ . Define  $v_{v,w}^{(2)}$  and  $a_{v,w}^{(2)}$  similarly with respect to the region enclosing volume  $w$ . Finally, let  $c_{v,w}$  denote the surface area of the separating cap. Then the surface area of the standard double bubble is

$$(58) \quad A_{H^3}(v + v_{v,w}^{(1)}) + A_{H^3}(w + v_{v,w}^{(2)}) - a_{v,w}^{(1)} - a_{v,w}^{(2)} + c_{v,w}.$$

Suppose that  $\theta$  is the angle between the separating cap and the disc with the same boundary, and let  $y$  be the radius of the disc. By Lemma 4.10, as  $v$  and  $w$  grow large,  $\theta \rightarrow 0$  and  $y \rightarrow y_0 = \cosh^{-1}(2)$ . Using formulas for volume and surface area, one can easily check that

$$\begin{aligned} \lim_{v,w \rightarrow \infty} v_{v,w}^{(1)} &= \lim_{v,w \rightarrow \infty} v_{v,w}^{(2)} \\ &= \pi \left( \frac{\sinh y_0 \sin(\pi/3)}{\operatorname{sech} y + \cos(\pi/3)} - \tanh^{-1} \left( \frac{\sinh y_0 \sin(\pi/3)}{\cos y + \cos(\pi/3)} \right) \right) \\ &= v_\infty \end{aligned}$$

$$\begin{aligned}
\lim_{v,w \rightarrow \infty} a_{v,w}^{(1)} &= \lim_{v,w \rightarrow \infty} a_{v,w}^{(2)} \\
&= \frac{2\pi \sinh^2 y_0}{1 + \cos(\pi/3) \cosh y_0} \\
&= a_\infty
\end{aligned}$$

and

$$\begin{aligned}
\lim_{v,w \rightarrow \infty} c_{v,w} &= \frac{2\pi \sinh^2 y_0}{1 + \cos 0 \cosh y_0} \\
&= c_\infty.
\end{aligned}$$

Hence,

$$|A(v + v_{v,w}^{(1)}) - A(v + v_\infty)| < A'(v)|v_{v,w}^{(1)} - v_\infty|$$

approaches 0 as  $v, w$  grow large. So too does  $|A(w + v_{v,w}^{(2)}) - A(w + v_\infty)|$ . From these computed limits, it follows that the surface area of the standard double bubble enclosing volumes  $v, w$  – as calculated in Equation 58 – approaches  $A(v + v_\infty) + A(w + v_\infty) - 2a_\infty + c_\infty$  as  $v, w$  get large.  $\square$

**Proposition 4.12.** *For each fixed  $\psi > 0$ ,*

$$\lim_{w \rightarrow \infty} F(\psi w, w) = 2\pi \ln \frac{4(\psi + 1)}{e^2}.$$

(Note that this limit is nonnegative if and only if  $\psi \geq \lambda$  which is less than 0.85.)

*Proof.* We write  $v = v(w) = \lambda w$ . Let  $v_\infty = \pi(3/2 - \ln 2)$ ,  $a_\infty = 3\pi$ , and  $c_\infty = 2\pi$ . By Lemma 4.11, the given limit equals

$$(59) \quad \lim_{w \rightarrow \infty} 2A\left(\frac{v}{2}\right) + A(w) + A(v + w) - 2(A(v + v_\infty) + A(w + v_\infty) - 2a_\infty + c_\infty)$$

provided that the latter limit exists. By Lemma 4.3,

$$(60) \quad \lim_{x \rightarrow \infty} A(x) = 2x + 2\pi \ln x - 2\pi(1 - \ln(\pi/2))$$

for  $x \in \{\frac{v}{2}, w, v + w, 2v + v_\infty, w + v_\infty\}$ . Substituting these expressions into 59 and simplifying gives

$$\begin{aligned}
&\lim_{w \rightarrow \infty} (2\pi(2 \ln(\frac{v}{2}) + \ln w + \ln(v + w) - 2 \ln(v + v_\infty) - 2 \ln(w + v_\infty)) \\
&\quad - 8v_\infty + 4a_\infty - 2c_\infty)
\end{aligned}$$

provided that this new limit exists. Indeed, this new limit does exist and equals

$$\begin{aligned}
&\lim_{w \rightarrow \infty} (2\pi \ln \left( \frac{v^2 w (v + w)}{4(v + v_\infty)^2 (w + v_\infty)^2} \right) - 8v_\infty + 4a_\infty - 2c_\infty) \\
&= \lim_{w \rightarrow \infty} (2\pi \ln \left( \frac{v + w}{4w} \right) - 8v_\infty + 4a_\infty - 2c_\infty) \\
&= 2\pi \ln \left( \frac{\psi + 1}{4} \right) - 8 \cdot \pi(3/2 - \ln 2) + 4 \cdot 3\pi - 2 \cdot 2\pi \\
&= 2\pi \ln \left( \frac{4(\psi + 1)}{e^2} \right).
\end{aligned}$$

Hence the limit in (59) evaluates to  $2\pi \ln(\frac{4(\psi+1)}{e^2})$ , which is only positive when  $\psi \geq \lambda$ , finishing the proof.  $\square$

#### 4.4. The Hutchings function is decreasing along a line for large volumes.

**Lemma 4.13.** *Given a standard double bubble enclosing volumes  $v, w$ , the outer cap of the region of volume  $v$  lies on a sphere, say with volume  $v + v_1$  (resp. the outer cap of the region of volume  $w$  lies on a sphere with volume  $w + w_1$ ). If  $v, w > 300\lambda$  then  $v_1, w_1 < 3$ .*

*Proof.* Let  $\text{vol}(y, \alpha)$  denote the volume of a spherical cap with associated disc of radius  $y$ , where the cap and disc meet at an angle  $[0, \pi]$ . This volume is increasing in both  $y$  and  $\alpha$ . Without loss of generality, assume that  $v \leq w$ . The three caps of the standard double bubble meet along a circular interface with radius  $y_0$ ; let the disc with this circle as its boundary meet the separating cap of the double bubble at angle  $\theta$ . By Lemma 4.8,  $y_0 < \cosh^{-1}(2)$ . By Lemma 4.10,  $\theta < 1/20$ . Both  $v_1 = \text{vol}(y_0, \pi/3 + \theta) - \text{vol}(y_0, \theta)$  and  $w_1 = \text{vol}(y_0, \pi/3 - \theta) + \text{vol}(y_0, \theta)$  are less than  $\text{vol}(y_0, \pi/3 + \theta) + \text{vol}(y_0, \theta)$ , which in turn is less than  $\text{vol}(\cosh^{-1}(2), \pi/3 + 1/20) + \text{vol}(\cosh^{-1}(2), 1/20) < 3$ .  $\square$

We now prove some algebraic lemmas that will be used in the proof of Proposition 4.20 and later in Proposition 4.21.

**Lemma 4.14.** *If  $\pi \ln x + a > 0$ , then  $\frac{1}{x + \pi \ln x + a} > \frac{1}{x} - \frac{\pi \ln x + a}{x^2}$ .*

*Proof.* The left hand side equals

$$(61) \quad \frac{1}{x} - \frac{\pi \ln x + a}{x(x + \pi \ln x + a)},$$

which is greater than  $\frac{1}{x} - \frac{\pi \ln x + a}{x^2}$ .  $\square$

**Corollary 4.15.** *For positive numbers  $\mu, w$  with  $\ln \mu + a > 0$*

$$(62) \quad \frac{\mu}{\mu w + \pi \ln(\mu w) + a} > \frac{1}{w} - \frac{1}{w^2} \left( \frac{\pi \ln w}{\mu} + \frac{\pi \ln \mu}{\mu} - \frac{a}{\mu} \right).$$

*Proof.* Simply apply Lemma 4.14 with  $x = \mu w$ .  $\square$

**Lemma 4.16.** *If  $x > 150\lambda$ , then*

$$\frac{1}{x + \pi \ln x - 3} < \frac{1}{x} - \frac{\pi \ln x - 3}{1.1x^2}.$$

*Proof.* For such  $x$ , we have  $1.1x > x + \pi \ln x - 3 > 0$ . Hence,

$$(63) \quad \frac{1}{x + \pi \ln x - 3} = \frac{1}{x} - \frac{\pi \ln x - 3}{x(x + \pi \ln x - 3)} < \frac{1}{x} - \frac{\pi \ln x - 3}{1.1x^2}$$

as desired.  $\square$

**Corollary 4.17.** *For positive  $\mu, w$ , with  $\mu w > 150$*

$$(64) \quad \frac{\mu}{\mu w + \pi \ln(\mu w) - 3} < \frac{1}{w} - \frac{1}{w^2} \left( \frac{\pi \ln w}{1.1\mu} + \frac{\pi \ln \mu}{1.1\mu} - \frac{3}{1.1\mu} \right).$$

*Proof.* Apply Lemma 4.16 with  $x = \mu w$ .  $\square$

**Lemma 4.18.** *For all  $w \geq 300$ ,*

$$\begin{aligned} & 2 \left( \frac{\pi \ln w}{\lambda} + \frac{\pi \ln \lambda}{\lambda} - \frac{2}{\lambda} + \pi \ln w - 2 \right) < \\ & \frac{2\pi \ln w}{1.1 \frac{\lambda}{2}} + \frac{2\pi \ln(\frac{\lambda}{2})}{1.1 \frac{\lambda}{2}} - \frac{6}{1.1 \frac{\lambda}{2}} + \frac{\pi \ln w}{1.1} - \frac{3}{1.1} + \\ & \frac{\pi \ln w}{1.1(\lambda+1)} + \frac{\pi \ln(\lambda+1)}{1.1(\lambda+1)} - \frac{3}{1.1(\lambda+1)}. \end{aligned}$$

*Proof.* For all  $w \geq 300$ ,

$$\begin{aligned} & \pi \left( 2 \frac{\ln \lambda}{\lambda} - 2 \frac{\ln \lambda/2}{1.1 \lambda/2} - \frac{\ln(\lambda+1)}{1.1(\lambda+1)} \right) \\ & + \left( 2 \frac{3}{1.1 \lambda/2} + \frac{3}{1.1} + \frac{3}{1.1(\lambda+1)} - \frac{4}{\lambda} - 4 \right) < \\ & \left( \frac{2}{1.1 \lambda/2} + \frac{1}{1.1} + \frac{1}{1.1(\lambda+1)} - \frac{2}{\lambda} - 2 \right) \pi \ln w. \end{aligned}$$

Simple algebraic manipulation gives the desired result.  $\square$

**Lemma 4.19.** *If  $w \geq 300$ , then*

$$\begin{aligned} & 4 + 4\lambda + 2\pi \left( \frac{2\lambda}{\lambda w + \pi \ln(\lambda w) + 2} + \frac{2}{w + \pi \ln(w) + 2} \right) \\ & > 4 + 4\lambda + 2\pi \left( \frac{\lambda}{\lambda \frac{w}{2} + \pi \ln(\lambda \frac{w}{2}) - 3} + \frac{1}{w + \pi \ln(w) - 3} \right. \\ & \quad \left. + \frac{\lambda + 1}{(\lambda + 1)w + \pi \ln((\lambda + 1)w) - 3} \right). \end{aligned}$$

*Proof.* By Corollary 4.15,

$$\begin{aligned} & \frac{2\lambda}{\lambda w + \pi \ln(\lambda w) + 2} + \frac{2}{w + \pi \ln(w) + 2} \\ & > 2 \left( \frac{2}{w} - \frac{1}{w^2} \left( \frac{\pi \ln w}{\lambda} + \frac{\pi \ln(\lambda)}{\lambda} - \frac{2}{\lambda} \right) - \frac{1}{w^2} (\pi \ln w - 2) \right) \\ & > \frac{4}{w} - \frac{1}{w^2} \left( \frac{2\pi \ln w}{1.1 \frac{\lambda}{2}} + \frac{2\pi \ln(\frac{\lambda}{2})}{1.1 \frac{\lambda}{2}} - \frac{6}{1.1 \frac{\lambda}{2}} + \frac{\pi \ln w}{1.1} - \frac{3}{1.1} \right. \\ & \quad \left. + \frac{\pi \ln w}{1.1(\lambda+1)} + \frac{\pi \ln(\lambda+1)}{1.1(\lambda+1)} - \frac{3}{1.1(\lambda+1)} \right) \end{aligned}$$

(by Lemma 4.18)

$$> \frac{\lambda}{\frac{\lambda w}{2} + \pi \ln \frac{\lambda w}{2} - 3} + \frac{1}{w + \pi \ln w - 3} + \frac{(1 + \lambda)}{(\lambda + 1)w + \pi \ln((\lambda + 1)w) - 3}$$

(by Corollary 4.17). Multiplying both sides of the inequality by  $2\pi$  and adding  $4(1 + \lambda)$  yields the desired result.  $\square$

**Proposition 4.20.** *If  $w \geq 300$ , then*

$$\frac{d}{dw} F(\lambda w, w) < 0.$$

*Proof.* Consider a standard double bubble enclosing volumes  $\lambda w$  and  $w$ . Let  $v_1, w_1$ , be the volumes needed to complete the two outer caps of the double bubble which enclose volumes  $\lambda w$  and  $w$ . Then,

$$(65) \quad \frac{dA}{dw}(\lambda w, w) = \lambda A'(\lambda w + v_1) + A'(w + w_1).$$

The bound on  $A'(x)$  given in Lemma 4.5 tells us that

$$(66) \quad \left( 2 + \frac{2\pi}{\mu w + \pi \ln(\mu w) - 3} \right) > A'(\mu w).$$

Hence,

$$(67) \quad 2\lambda + 2\pi \left( \frac{\lambda}{\lambda \frac{w}{2} + \pi \ln(\lambda \frac{w}{2}) - 3} \right) > \lambda A' \left( \frac{\lambda w}{2} \right),$$

$$(68) \quad 2 + 2\pi \left( \frac{1}{w + \pi \ln(w) - 3} \right) > A'(w),$$

$$(69) \quad 2 + 2\pi \left( \frac{\lambda + 1}{(\lambda + 1)w + \pi \ln((\lambda + 1)w) - 3} \right) > A'((\lambda + 1)w).$$

By Corollary 4.7,

$$(70) \quad \lambda A'(\lambda w + 3) + A'(w + 3) > \lambda \left( 2 + \frac{2\pi}{\lambda w + \pi \ln(\lambda w) + 2} \right) + \left( 2 + \frac{2\pi}{w + \pi \ln w + 2} \right).$$

Since  $\lambda w$  and  $w$  are greater than  $300\lambda$ , we can apply Lemma 4.13 to see that  $v_1, w_1 < 3$ . Hence,

$$(71) \quad \lambda A'(\lambda w + v_1) + A'(w + w_1) > \lambda A'(\lambda w + 3) + A'(w + 3).$$

Therefore,

$$(72) \quad \lambda A'(\lambda w + v_1) + A'(w + w_1) > \lambda \left( 2 + \frac{2\pi}{\lambda w + \pi \ln \lambda w + 2} \right) + \left( 2 + \frac{2\pi}{w + \pi \ln w + 2} \right).$$

Combining inequalities 72, 67, 68, 69 and Proposition 4.19, we can see for all  $\lambda w, w > 300\lambda$ ,

$$(73) \quad 2\lambda A'(\lambda w + v_1) + 2A'(w + w_1) > \lambda A' \left( \frac{\lambda w}{2} \right) + A'(w) + A'(\lambda w + w).$$

Thus,

$$(74) \quad 2 \frac{d}{dw} A(\lambda w, w) > \lambda A' \left( \frac{\lambda w}{2} \right) + A'(w) + A'(\lambda w + w)$$

as desired. □

#### 4.5. The Hutchings function is increasing in $v$ for large volumes.

**Proposition 4.21.** *For  $v \geq \lambda w$ ,  $w \geq 150$ ,*

$$(75) \quad \frac{1}{\frac{v}{2} + \pi \ln \frac{v}{2} - 1.041} + \frac{1}{v + w + \pi \ln(v + w) - 1.041} > 2 \frac{1}{v \pi \ln v - 3}$$

*Proof.* Since  $v \geq \lambda w$ ,  $w \geq 150$ ,

$$(76) \quad \frac{v^2}{2} > 4(v + w)(\pi \ln \frac{v}{2} - 1.041)$$

and

$$(77) \quad \frac{v + w}{2} > \pi \ln(v + w).$$

Hence,

$$(78) \quad v^2(v + w) > (v + w)^2 \frac{\pi \ln \frac{v}{2} - 1.041}{\frac{1}{4}} + v^2 \pi \ln(v + w) - 1.041v^2 - (v + w)^2 \frac{\pi \ln v - 3}{1.1}.$$

So,

$$(79) \quad \frac{2}{v} - \frac{\pi \ln \frac{v}{2} - 1.041}{(\frac{v}{2})^2} + \frac{1}{v + w} - \frac{\pi \ln(v + w) - 1.041}{(v + w)^2} > 2 \left( \frac{1}{v} - \frac{\pi \ln v - 3}{1.1v^2} \right).$$

Applying Lemmas 4.14 and 4.16 to the above equation yields,

$$(80) \quad \frac{1}{\frac{v}{2} + \pi \ln \frac{v}{2} - 1.041} + \frac{1}{v + w + \pi \ln(v + w) - 1.041} > 2 \frac{1}{v + \pi \ln v - 3}$$

as desired. □

**Lemma 4.22.** *For any fixed  $w \geq 150$  and  $v \geq \lambda w$ ,*

$$(81) \quad \frac{\partial F}{\partial v} > 0,$$

*Proof.* Differentiating the Hutchings function with respect to  $v$  yields

$$(82) \quad \frac{\partial F}{\partial v} = A'(\frac{v}{2}) + A'(v + w) - 2 \frac{\partial}{\partial v} A(v, w) > A'(\frac{v}{2}) + A'(v + w) - 2A'(v).$$

By Lemma 4.6 and Lemma 4.5,

$$(83) \quad \frac{\partial F}{\partial v} > \frac{1}{\frac{v}{2} + \pi \ln \frac{v}{2} - 1.041} + \frac{1}{v + w + \pi \ln(v + w) - 1.041} - 2 \frac{1}{v + \pi \ln v - 3}.$$

By Proposition 4.21,

$$(84) \quad \frac{\partial F}{\partial v} > 0,$$

as desired. □

#### 4.6. Conclusion.

**Theorem 4.23** (Hutchings function is positive for large volumes). *For all  $v \leq w$  such that  $w > 300$  and  $v > .85w$  (or indeed  $v \geq \lambda w \approx .841w$ ), the Hutchings function  $F(v, w) = 2A(\frac{v}{2}) + A(w) + A(v + w) - 2A(v, w)$  is positive.*

*Proof.* Consider  $(v, w) = (v_0, w_0)$ , where  $\lambda = v_0/w_0 > .85 > \lambda$  and  $w_0 > 300$ . By Proposition 4.22 the derivative of  $F$  with respect to its first argument is positive for the volume pairs under consideration, so  $F(v_0, w_0) > F(\lambda w_0, w_0)$ .

By Proposition 4.20  $F(\lambda w, w)$  decreases as  $w$  increases and by Proposition 4.12,

$$F(\lambda w_0, w_0) > \lim_{w \rightarrow \infty} F(\lambda w, w) = 0.$$

Hence,  $F(v_0, w_0) > 0$ , as claimed. □

**Remark 4.24.** The ratio  $\lambda = e^2/4 - 1$  is sharp in the following sense: for  $\psi < \lambda$ , there exist arbitrarily large  $v, w$  with  $v/w = \psi$  such that the modified Hutchings function is negative. This is a result of Proposition 4.12. (In fact, if the standard double bubble is the minimizer for all volume pairs – that is, if the modified Hutchings function equals the Hutchings function – then the ratio  $\lambda$  is the best we can do.)

## 5. THE COMPUTER PROOF OF THE POSITIVITY OF THE HUTCHINGS FUNCTION IN $S^3$ AND $H^3$ .

**5.1. Introduction.** Rigorously showing positivity of the Hutchings function is equivalent to showing that  $g(v, w) = 2A(\frac{v}{2}) + A(w) + A(v + w)$  is strictly greater than  $h(v, w) = 2A(v, w)$ . The functions  $g$  and  $h$  have very similar properties in  $S^3$  and  $H^3$ , but the proofs vary slightly because the area of single bubbles and standard double bubbles are not increasing for all volumes in  $S^3$  and the  $H^3$  code has to efficiently calculate the curvatures of double bubbles.

The computer proof relies on the facts that  $g$  is concave (Proposition 2.15) and  $h$  is increasing on relevant domains (Propositions 2.16 and 2.17) in both spaces and that  $g$  is increasing on all of  $H^3$ . If  $g$  and  $h$  were each concave increasing functions of one variable, we could show that  $g$  is bigger than  $h$  on the interval  $[v, v + b]$ , by showing  $g(v) > h(v + a)$  and  $g(v + a) > h(v + b)$  (See Figure 4). The computational lemma (Lemma 5.1) extends this idea into two dimensions showing that  $g > h$  on a polygonal domain in the  $vw$ -plane if the minimum value of  $g$  on the vertex set is bigger than the maximum value of  $h$ .

Using a computer does introduce error, which must be accounted for. A computer uses approximations for  $g$  and  $h$ , which we will call  $g_{comp}$  and  $h_{comp}$ . However if  $g$  is underestimated and  $h$  is overestimated then the computer test will show  $g_{comp} > h_{comp}$  only if  $g > h$ .

The following lemma shows that checking a finite number of points can give a global inequality.

**Lemma 5.1.** *Given a polygonal domain  $D$  with vertices identified by ordered pairs  $(v_i, w_i)$ , a concave function  $g(v, w)$ , and another function  $h(v, w)$  that*

obtains its maximum at  $(v_0, w_0)$ , if  $\min(g(v_i, w_i)) > h(v_0, w_0)$  then  $g > h$  on  $D$ .

*Proof.* Since  $g$  is concave it has a minimum on  $D$  at a vertex of the polygon. Thus,  $\min(g(v_i, w_i))$  is a lower bound for  $g$  on  $D$  and  $h(v_0, w_0)$  is an upper bound for  $h$  on  $D$ . Hence,  $g > h$  follows directly.  $\square$

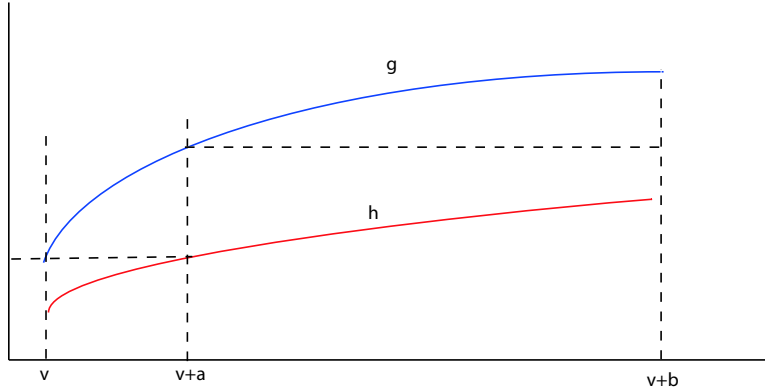


FIGURE 4. The steps to show that  $g > h$  if both are concave increasing functions of one variable.

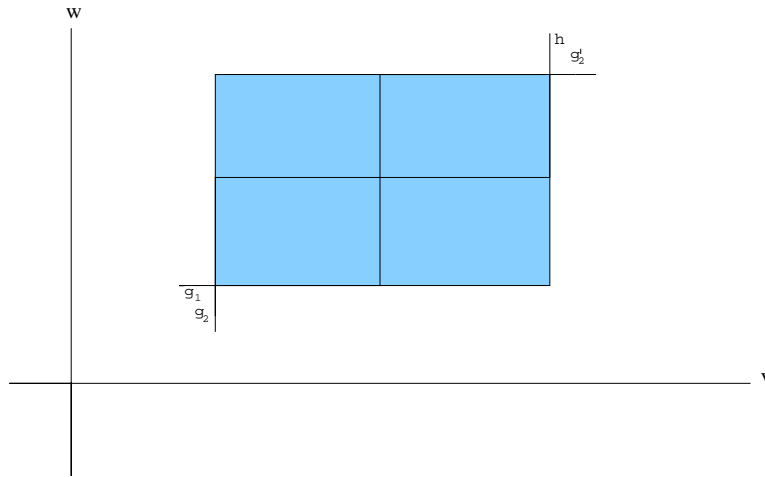


FIGURE 5. If  $g = g_1 + \min(g_2, g_2') > h$  and  $g_1$  and  $h$  are increasing on the rectangle, then the Hutchings function is positive on the shaded rectangle. If  $g \not> h$ , then the rectangle is split into the four smaller rectangles and the function checks to see if  $g > h$  on these smaller domains.

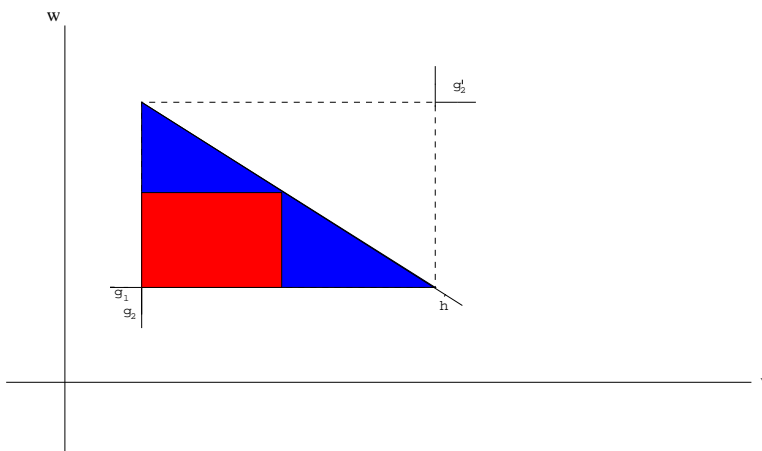


FIGURE 6. If  $g = g_1 + \min(g_2, g'_2) > h$ ,  $g_1$  is increasing on the triangle, and  $h$  is increasing along the hypotenuse of the triangle, then the Hutchings function is positive on the shaded triangle. If  $g < h$ , then the triangle is split into the two smaller triangles and the rectangle (See Figure 5) and the function checks to see if  $g > h$  on these smaller domains.

**The computer proof in  $S^3$ .** In  $S^3$ , rectangular regions and triangular regions are considered. There is additional computational complexity in  $S^3$  because  $A(v)$  and  $A(v, w)$  are not increasing for all volume pairs  $(v, w)$  in  $S^3$ . For rectangles where  $h(v, w)$  is strictly increasing in a neighborhood of the rectangle, the area of the standard double bubble is over-approximated by using radii of the outer spherical caps corresponding to volumes bigger than  $v$  and  $w$ . For the area of the single bubble each part of  $g$  is taken at its minimum.  $A(\frac{v}{2})$  and  $A(w)$  are always under-approximated by taking radii that correspond to spheres enclosing volumes less than  $\frac{v}{2}$  and  $w$ . Their sum represents  $g_1$  in Figures 5 and 6.  $A(v + w)$  is under-approximated by considering the minimum of the value of  $A$  at a point to the lower left and a point to the upper right of the rectangle. This is the minimum of  $g_2$  and  $g'_2$  in Figures 5 and 6.

Next, we consider the entire region where numerical analysis is necessary. The triangular domain is first broken into parts: the right triangle  $(\bar{v}, \bar{w}) = (.1, .1), (.1, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3})$  and the triangle  $(\bar{v}, \bar{w}) = (.1, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3})$  and  $(.1, .45)$ . The latter, upper triangle is difficult to deal with because  $h$  is increasing up to the hypotenuse and decreasing above the hypotenuse. We can assure that we land exactly on this line by setting the two parameters  $r_1$  and  $r_3$  to be equal. Getting an over-approximation for area on this line just involves overestimating  $r_1$ . In addition,  $A(\frac{1}{3}, \frac{1}{3})$  must be hard coded into the program. If we want to over-approximate  $h$  we move the point  $(v, w)$  southeast along the line where  $w = v$  and use that for our value of  $h$ . For  $g$ , we still just take the appropriate corners of the rectangle containing the triangle. If not, then we break the triangle into a rectangle and two smaller right triangles by connecting a point on the hypotenuse to the two legs of the triangle via perpendicular lines (See Figure 6). The desired inequality is then shown to be true

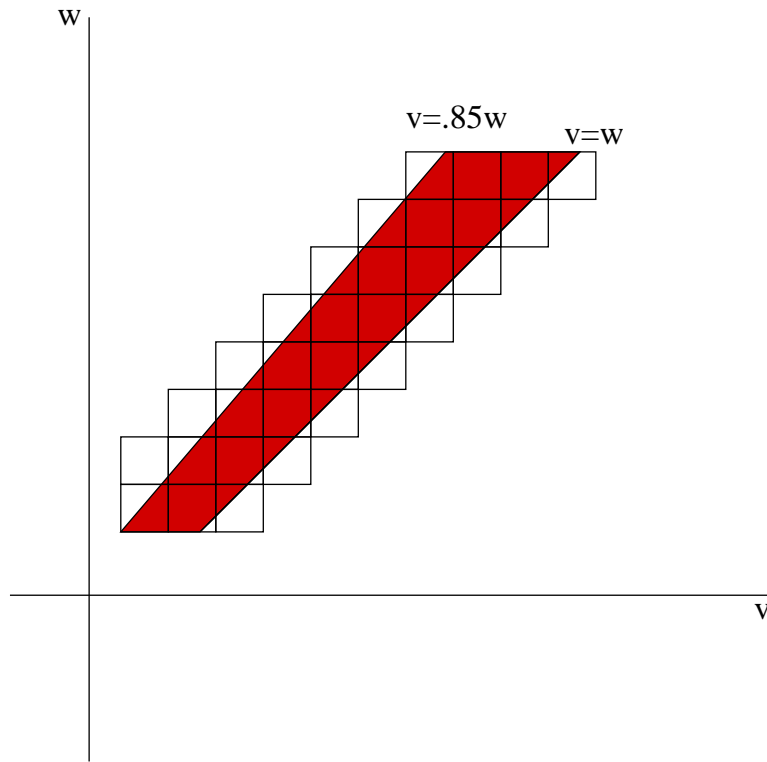


FIGURE 7. Showing that the Hutchings function is positive on the rectangles covering the shaded region shows that the function is positive on the shaded region. Also, showing that the function is positive on rectangles containing the line  $v=.85w$  shows that the Hutchings function is positive for this line.

on the rectangle and the two smaller triangles. Similarly for the rectangle if the desired inequality can not be shown on the large rectangle, the rectangle is split into four smaller rectangles and the inequality is shown to hold true on them (See Figure 5). Both subdivisions repeat until the desired inequality can be shown on each subdivided rectangle or triangle.

The lower triangle can be replaced by the rectangle containing it, which can then be subdivided into smaller rectangles by the argument above.

Mathematica uses computer algebra (with infinite precision variables). When checking inequalities, it reports true, false, or null if it cannot tell for sure [W].

**The computer proof in  $\mathbf{H}^3$ .** In  $\mathbf{H}^3$ , rectangular regions that tile the area between two lines through the origin are considered (See Figure 7). The function  $g$  is strictly increasing, so  $g(v_0, w_0)$  is less than any value of  $g(v, w)$  on a rectangular region with  $v_0 \leq v$  and  $w_0 \leq w$ . We can assure that these two conditions hold if we take a guess at the curvature that we know to be too low (corresponding to a

volume that is too high). Then, we increase the curvature (decreasing the volume) of the sphere by multiplying it by some constant. This process is done first with a relatively large constant until the curvature corresponds to volume that is too low; then the curvature is divided by the constant and the curvature is increased by a smaller constant until it corresponds to a volume that is less than the volume being approximated. Using this curvature for our surface area calculations under-approximates the surface area of single bubbles.

For  $A(v, w)$ , the double bubble is completely determined by the mean curvatures of the outer caps. Thus, under-approximating the curvatures of these caps (by an analogous method to the single bubble case) corresponds to a double bubble with larger volumes and hence more surface area.

We use Mathematica's built-in *SetAccuracy* function, which assures accuracy on 25 binary digits right of the decimal point in our important computations. When we calculate curvature, we check that the calculated associated volume  $VolSphere[k]$  is less than  $v - 2^{-24}$ , which guarantees that the associated volume is less than  $v$ . Hence the associated area will be less than  $A$ . Finally, we subtract  $2^{-24}$  from our calculated area to be sure that we have a lower bound.

We check that the calculated associated volumes  $VolBubV[k1, k2]$  and  $VolBubW[k1, k2]$  exceed  $v$  and  $w$  by  $2^{-23} = 2 \cdot 2^{-24}$ , because sometimes there are two numbers summed in the calculation. Hence the associated area will be greater than  $A$ . We add  $3 \cdot 2^{-24}$  to the calculated area to be sure.

For Claim 5.21 the proof-function is restricted to showing that the Hutchings function is positive only on rectangles that include the line  $v = .85w$ .

**Remark 5.2.** In  $S^3$ , the proof-function determines the size of each polygon via a recursive algorithm. Both proof-functions (Claims 5.5 and 5.6) complete in well under six hours each depending on computer speed. In  $H^3$ , the proof-functions (Claims 5.10-5.20) take well over 150 hours to complete. The function is broken into pieces in order to maximize efficiency by choosing the rectangle size to be relatively large on a given region. Finally, the claim dealing with the ray  $v = .85w$  (Claim 5.21) completes in under three hours.

**5.2. Program implementation.** This section discusses the code in the appendix. The computer proof in  $S^3$  uses two proof functions. The first is called ProofFunctionTriangle. The other is called ProofFunctionRectangle. The following claims can be proved by examining and running the code in the Appendix (Section 7). The proof-functions returns a 1 if the Hutchings function is positive on the rectangle or triangle it is called upon.

Implementation in  $S^3$ .

**Claim 5.3.** *The computed function  $A(v, error_)$  returns a lower bound on the area of a sphere in  $S^3$  with given volume  $v$ .*

**Claim 5.4.** *The function  $A(v, w, VError_, WError_, changeToV_, changeToW_)$  gives an upper bound on  $A(v, w)$ .*

**Claim 5.5.** *If ProofFunctionRectangle returns a 1 on a region where  $A(v)$  and  $A(v,w)$  are increasing in  $v,w$ , then the Hutchings Function is positive on this region.*

**Claim 5.6.** *If ProofFunctionTriangle returns a 1 on a region when  $A(v)$  and  $A(v,w)$  are increasing in  $v,w$  and on the hypotenuse of the triangle, then the Hutchings Function is positive on this region.*

**Claim 5.7.** *ProofFunctionRectangle[lhsF2, rhsF2, {VolOfS3/10, VolOfS3/10}, {VolOfS3/3, VolOfS3/3}, lhsF2[VolOfS3/10, VolOfS3/10], rhsF2[VolOfS3/10, VolOfS3/10], lhsF2[VolOfS3/3, VolOfS3/3], rhsF2[VolOfS3/3, VolOfS3/3], temp] returns a 1 where*

$$\begin{aligned} lhsF2 &= 2 * AreaSphereGivenVolume[v/2, VError/2] + \\ &\quad AreaSphereGivenVolume[w, WError] + \\ &\quad AreaSphereGivenVolume[v + w, VError] \end{aligned}$$

and

$$rhsF2 = 2 * A[v, w, VError, WError, ChangeInV, ChangeInW].$$

**Claim 5.8.** *ProofFunctionTriangle[lhsF2, rhsF2, VolOfS3/10, VolOfS3/3, VolOfS3/3, 9\*VolOfS3/20, lhsF2[VolOfS3/10, VolOfS3/3], rhsF2[VolOfS3/10, VolOfS3/3], lhsF2[VolOfS3/10, 9\*VolOfS3/20], rhsF2[VolOfS3/10, 9\*VolOfS3/20], lhsF2[VolOfS3/3, VolOfS3/3], rhsF2[VolOfS3/3, VolOfS3/3], temp] returns a 1 where*

$$\begin{aligned} lhsF2 &= 2 * AreaSphereGivenVolume[v/2, VError/2] + \\ &\quad AreaSphereGivenVolume[w, WError] + \\ &\quad AreaSphereGivenVolume[v + w, VError] \end{aligned}$$

and

$$rhsF2 = 2 * A[v, w, VError, WError, ChangeInV, ChangeInW].$$

Implementation in  $\mathbf{H}^3$ . The following claims can be proved by examining and running the code in the appendix. The program stops if it finds a rectangle where  $g \leq h$ . Therefore, if the proof-function completes, the Hutchings function is positive on the region it was called on (Claim 5.9).

**Claim 5.9.** *If ArrayFillingProof completes then the Hutchings function is positive on the region it was called on.*

**Claim 5.10.** *ArrayFillingProof[.002329, .00274, .01, .00001, .00001, 11.46, 10.95, .9999, .9995] completes.*

**Claim 5.11.** *ArrayFillingProof[.0085, .01, .1, .00005, .00005, 7.475, 7.15, .9999, .9995] completes.*

**Claim 5.12.** *ArrayFillingProof[.085, .1, 1, .0005, .0005, 3.56, 3.415, .9999, .9995] completes.*

**Claim 5.13.** *ArrayFillingProof*[.85, 1., 15., .005, .005, 1.849, 1.787, .9999, .9995] completes.

**Claim 5.14.** *ArrayFillingProof*[12.75, 15., 25., .01, .01, 1.15204, 1.1352, .9999, .9995] completes.

**Claim 5.15.** *ArrayFillingProof*[21.25, 25., 45., .02, .02, 1.1027, 1.09054, .9999, .9995] completes.

**Claim 5.16.** *ArrayFillingProof*[38.25, 45., 65., .02, .02, 1.0637, 1.05562, .9999, .9995] completes.

**Claim 5.17.** *ArrayFillingProof*[55.25, 65., 85., .02, .02, 1.04658, 1.040469, .9999, .9995] completes.

**Claim 5.18.** *ArrayFillingProof*[72.25, 85., 110., .015, .015, 1.03684, 1.031905, .9999, .9995] completes.

**Claim 5.19.** *ArrayFillingProof*[93.5, 110., 130., .015, .015, 1.02927, 1.025276, .9999, .9995] completes.

**Claim 5.20.** *ArrayFillingProof*[110.5, 130., 150., .015, .015, 1.025161, 1.021693, .9999, .9995] completes.

Finally if we adjust the *ArrayFillingProof*, so that it only shows that the Hutchings function is positive along the line  $v = .85w$ ,

**Claim 5.21.** *ArrayFillingProof*[127.5, 150., 300., .01, .01, 1.022077, 1.019009, .9999, .9995] completes.

**5.3. Main propositions.** The following propositions consolidate the claims that the computer code runs showing the the Hutchings function is positive on the desired domains. Proposition 5.24 uses a modified version of the code that checks only that the Hutchings function is positive on a small domain that includes the line  $v = .85w$ .

**Proposition 5.22.** *If an area-minimizing double bubble in  $S^3$  encloses volumes  $.1 \leq \bar{v} \leq \min\{\bar{w}, 1 - 2\bar{w}\}$ , then the Hutchings function  $F_{S^3}(v, w) > 0$ .*

*Proof.* This follows from implementing the code in the appendix (Claims 5.7, 5.8), and Claims 5.5 and 5.6.  $\square$

**Proposition 5.23.** *If an area-minimizing double bubble in  $H^3$  encloses volumes  $v, w$  such that  $.002743 \leq v \leq w \leq 150$  and  $v \geq .85w$  then the Hutchings function  $F_{H^3}(v, w) > 0$ .*

*Proof.* This follows from Claim 5.9 and Claims 5.10-5.20.  $\square$

**Proposition 5.24.** *The Hutchings function  $F_{H^3}(v, w) > 0$ , for volume pairs  $(v, w)$ , where  $.85v = w$  and  $w \in [150, 300]$ .*

*Proof.* This follows from Claim 5.9 and Claim 5.21.  $\square$

6. DOUBLE BUBBLES IN  $\mathbf{S}^3$  AND  $\mathbf{H}^3$ 

Theorems 6.3 and 6.4 provide our main results on double bubbles in  $\mathbf{S}^3$  and  $\mathbf{H}^3$ . They depend on Propositions 6.1 and 6.2 of Cotton and Freeman [CF], after Hutchings, Morgan, Ritoré, and Ros [HMRR], which reduce the proofs to our connectivity results of sections 3-5 via the Hutchings Function.

**Proposition 6.1.** *An area-minimizing double bubble in  $\mathbf{S}^3$  for which both enclosed regions and the exterior are connected must be standard.*

*Proof.* See Cotton and Freeman [CF, Proposition 7.3]. □

**Proposition 6.2.** *An area-minimizing double bubble in  $\mathbf{H}^3$  for which both enclosed regions are connected must be standard.*

*Proof.* See Cotton and Freeman [CF, Proposition 7.7]. □

**Theorem 6.3** (Area-minimizing double bubbles in  $\mathbf{S}^3$ ). *An area-minimizing double bubble in  $\mathbf{S}^3$  is standard if the volume of each of the three regions is larger than 10% of the total volume of  $\mathbf{S}^3$ .*

*Proof.* By Proposition 6.1, it suffices to show that all regions are connected. By Lemma 2.19, we know that the region of volume  $v$  is connected whenever  $v > 2w$  or  $v > 2u$  (Figure 8 a). This reduces the problem to showing that the region of volume  $v$  is connected on the pentagonal domain of Figure 8 e defined by the points:  $(\bar{v}, \bar{w}, \bar{u}) = (.1, .1, .8), (.1, .8, .1), (.2, .7, .1), (.5, .25, .25)$ , and  $(.2, .1, .7)$ . By Proposition 2.6, it suffices to show that the Hutchings Function  $F(v, w)$  is positive on this domain. When  $.1 \leq \bar{v} \leq \bar{w}$  and  $\bar{v} \leq 1 - 2\bar{w}$ , computer analysis (Proposition 5.22, see Figure 8b) tells us that the Hutchings Function  $F_{\mathbf{S}^3}(v, w)$  is positive. In particular, for  $\bar{v} \leq \frac{1}{3}$ ,  $F_{\mathbf{S}^3}(v, v)$  is positive. When  $\frac{1}{3} \geq \bar{v} > \bar{w}$  and  $v \leq 2w$ , by balancing (Proposition 2.18, see Figure 8c)  $F_{\mathbf{S}^3}(v, w)$  is positive on the quadrilateral domain with endpoints:  $(\bar{v}, \bar{w}, \bar{u}) = (.1, .1, .8), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{4}{9}, \frac{2}{9}, \frac{1}{3})$ , and  $(.2, .1, .7)$ . Thus  $F(v, w) > 0$  on the pentagonal domain gained by balancing in addition to the region where the computer analysis was done. Hence by permutation (Proposition 2.20, see Figure 8d),  $F(v, w) > 0$  on the pentagonal domain with endpoints:  $(\bar{v}, \bar{w}, \bar{u}) = (.2, .7, .1), (\frac{4}{9}, \frac{1}{3}, \frac{2}{9}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (.1, .45, .45)$  and  $(.1, .8, .1)$ . In particular, if  $\bar{v} \leq .45$ ,  $F(v, v)$  is positive. By balancing (Proposition 2.18, see Figure 8e)  $F(v, w)$  is positive whenever  $\bar{w} \leq \bar{v} \leq 2\bar{w}$ . Thus,  $F(v, w)$  is positive, for on the pentagonal domain  $(\bar{v}, \bar{w}, \bar{u}) = (.1, .1, .8), (.1, .8, .1), (.2, .7, .1), (.5, .25, .25)$ , and  $(.2, .1, .7)$ . □

**Theorem 6.4** (Area-minimizing double bubbles in  $\mathbf{H}^3$ ). *An area-minimizing double bubble in  $\mathbf{H}^3$  is standard if the volume of the smaller region is at least 85% of the volume of the larger region.*

*Proof.* By Proposition 6.2 it suffices to show that the two enclosed regions are connected. By Proposition 2.6, it suffices to show that the Hutchings Function  $F(v, w)$  is positive for  $v \geq .85w$  (see Figure 9). Computer analysis (Propositions 5.23, 5.24) shows that when  $.002743 \leq v \leq w \leq 150$  and  $v \geq .85w$ , the Hutchings Function  $F_{\mathbf{H}^3}(v, w)$  is positive. It also shows that the Hutchings function is positive along the line segment  $v = .85w, w \in [150, 300]$ . Since the partial derivative of the Hutchings function with respect to  $v$  is positive for  $w \geq 150$  (Lemma 4.22),  $F(v, w) > 0$  for  $w \in [150, 300]$ . Asymptotic analysis (Theorems 3.5 and 4.23) shows

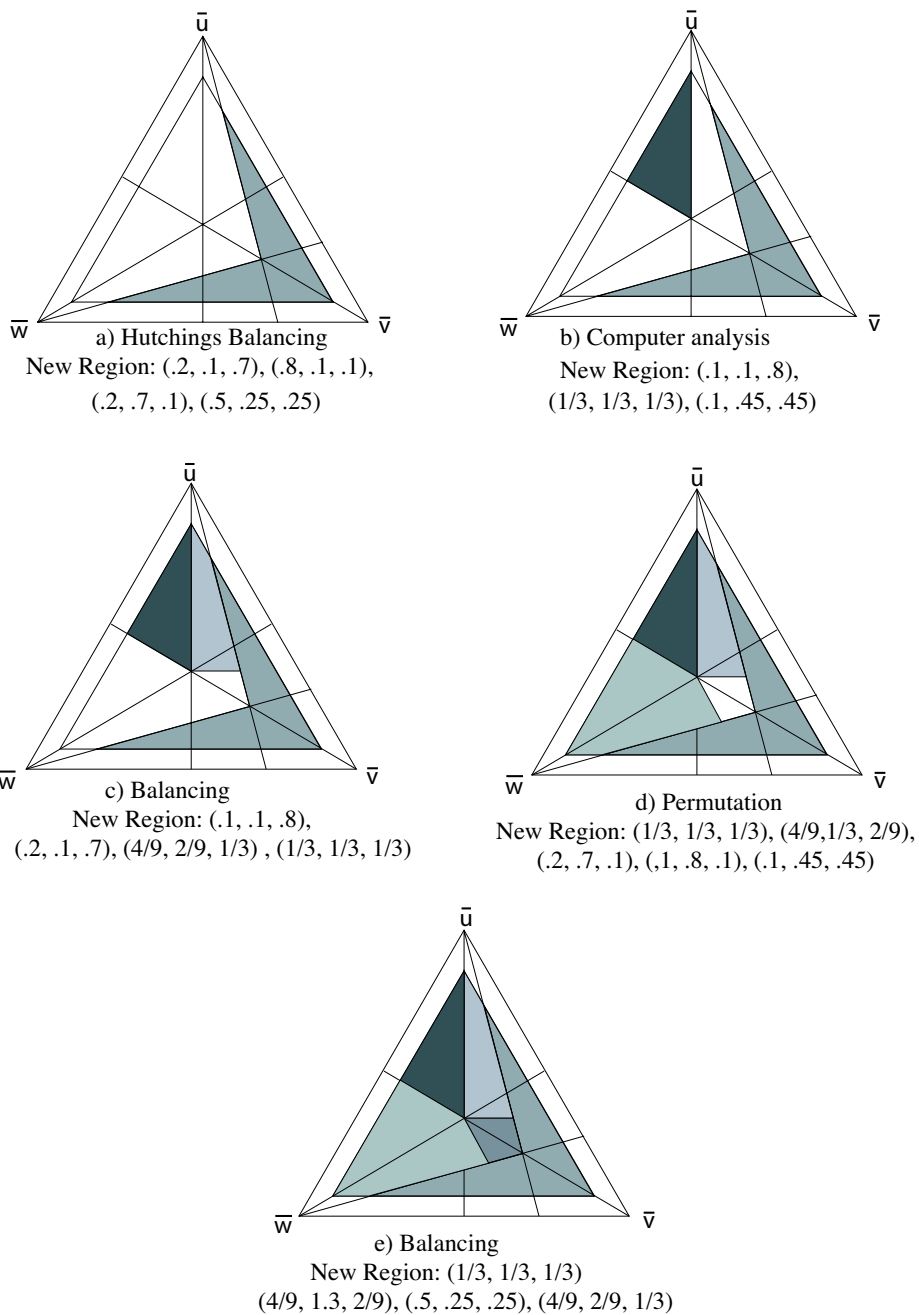


FIGURE 8. An illustration of the steps taken to cover the whole domain of volumes on  $S^3$ .

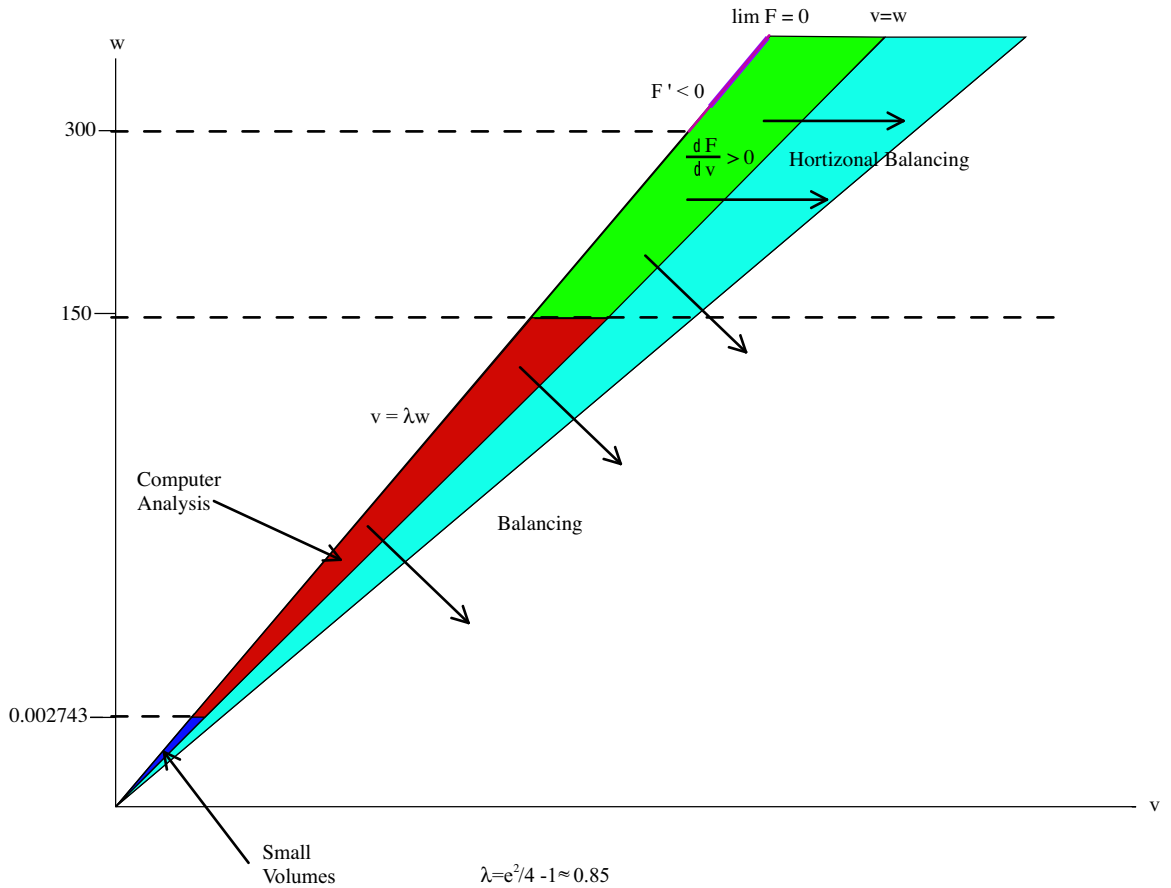


FIGURE 9. The basic steps of the proof that  $F$  is positive for  $v \geq .85w$  and  $w \geq .85v$  in  $\mathbf{H}^3$ .

that  $F_{\mathbf{H}^3}(v, w)$  is positive when  $v \leq w < .002743$  or  $v/.85 \geq w \geq 300$ . In particular,  $F_{\mathbf{H}^3}(v, v)$  is positive. Hence when  $v \geq w$ , Proposition 2.18 shows that  $F_{\mathbf{H}^3}(v, w)$  is positive.  $\square$

**Remark 6.5.** In hyperbolic quotient spaces, Theorem 6.4 and [CCWB, Remark 4.4] prove that the standard double bubble is area minimizing for some small volume pairs where the smaller volume is at least 85 percent that of the larger.

Future work. It follows from Theorem 6.4 and balancing (Lemma 2.19, Proposition 2.18) that the largest region in  $\mathbf{H}^3$  is always connected. Connectivity of the largest region is the key hypothesis for the even more general instability argument used in

$\mathbb{R}^4$  [Rei, Section 8]; future work might produce an analog of this result for hyperbolic space.

## 7. APPENDIX: PROGRAM CODE

### 7.1. $S^3$ code.

```

%NumDecimalPlacesGoal=10;
%SmallNumber=10^(-NumDecimalPlacesGoal);
VolOfS3=Pi*(2*Pi-Sin[2*Pi]);
%ChangeInV = 125/100;
%ChangeInW= 125/100;
%VError = 1/10000;
%WError = 1/10000;

ThetaFN[r1_,r2_] := ArcTan[Sqrt[3]*(Tan[r2]-Tan[r1])/(Tan[r1]+Tan[r2])];
xFN[r1_,r2_] := ArcTan[Tan[r1]*(Sqrt[3]/2*Cos[ThetaFN[r1,r2]]+1/2*
Sin[ThetaFN[r1,r2]])];
r3FN[r1_,r2_] := ArcCot[Cot[r1]-Cot[r2]];
Phi1FN[r1_,r2_] := ArcSin[Sin[xFN[r1,r2]]/Sin[r1]];
Psi1FN[r1_,r2_] := Module[{Phi1}, Phi1=Phi1FN[r1,r2]; If[(Tan[r2]-Tan[r1])/(
Tan[r1]+Tan[r2])-1/3<0, Pi-Phi1, Phi1]]
Phi2FN[r1_,r2_] := ArcSin[Sin[xFN[r1,r2]]/Sin[r2]];
Phi3FN[r1_,r2_] := ArcSin[Sin[xFN[r1,r2]]/Sin[r3FN[r1,r2]]];

VolCap[r_,Phi_] := If[r\Equal]Pi/2, (*
then*)0, (*else*)Pi (r-ArcTan[Cos[Phi]*Tan[r]]-Cos[r]*Sin[r]+Cos[
Phi]*Cos[r]*Sin[r]);

AreaCap[r_,Phi_] := 2Pi*(Sin[r]^2)*(1+Cos[Phi]);

VolSphere[r_] := Pi*(2*r-Sin[2*r]);

AreaSphereGivenRadius[r_] := 4*Pi*(Sin[r])^2

VolOneSDB[r1_,r2_] := Module[{Psi1,r3,Phi3}, Psi1=Psi1FN[r1,r2];
r3=r3FN[r1,r2];
Phi3=Phi3FN[r1,r2];
(*return*)VolCap[r1,Psi1]+VolCap[r3,Phi3]]

VolTwoSDB[r1_,r2_] := Module[{Phi2,r3,Phi3}, Phi2=Phi2FN[r1,r2];
r3=r3FN[r1,r2];
Phi3=Phi3FN[r1,r2];
(*return*)VolCap[r2, Pi-Phi2]-VolCap[r3,Phi3]]

EqualVolGivenRadius[r_] := Pi/2*(
2r-Sin[2r])* (1+(Sqrt[2]*Cos[r])/(Sqrt[7+Cos[2r]]))+Pi (ArcTan[Sqrt[2]Sin[r]\
/(Sqrt[7+Cos[2r]])]- (Sqrt[2]*r*Cos[r])/(Sqrt[7+Cos[2r]]));

AreaSDBGivenRadii[r1_,r2_] := Module[{Psi1,Phi2,Phi3,r3}, Psi1=Psi1FN[r1,r2];
Phi2=Phi2FN[r1,r2];
Phi3=Phi3FN[r1,r2];
r3=r3FN[r1,r2];
(*return*)AreaCap[r1, Pi-Psi1]+AreaCap[r2,Phi2]+AreaCap[r3, Pi-Phi3]]

RadiusSphere[v_, error_] := Module[{minRadius, maxRadius, currentRadius,
counter},

counter = 0;
minRadius = 0;
maxRadius = Pi;
currentRadius = Pi/2;

If[v VolOfS3/2,
While[VolSphere[currentRadius] > v || (VolSphere[currentRadius] < v \
-error),

If[VolSphere[currentRadius] > v,
maxRadius = currentRadius;
currentRadius = (currentRadius + minRadius)/2;

];

```

```

If[VolSphere[currentRadius] < v-error,

  minRadius= currentRadius;
  currentRadius = (currentRadius + maxRadius)/2;

  ];

counter++;

],

(*else if v > volofs3/2*)
While[VolSphere[currentRadius] < v || (VolSphere[currentRadius] > v \
+error),

  If[VolSphere[currentRadius] < v,

    minRadius= currentRadius;
    currentRadius = (currentRadius + maxRadius)/2;

    ];

  If[VolSphere[currentRadius] > v + error,

    maxRadius= currentRadius;
    currentRadius = (currentRadius + minRadius)/2;

    ];

  counter++;

  ];

];

(*Print["It took ", counter, "steps."];
Print["Radius is ", N[currentRadius]];*)
currentRadius
];

AreaSphereGivenVolume[v_, error_] := AreaSphereGivenRadius[RadiusSphere[v, \
error]];

RadiiSDB[v_, w_, vError_, wError_, vChangingFactor_,
wChangingFactor_] := Module[{VRadius, WRadius, madeChangeToV, \
madeChangeToW, adjustedVError,
adjustedWError, vAdjustment, wAdjustment, volOne, volTwo, counter},

  VRadius = Pi/4;
  WRadius = Pi/4;
  counter=0;

  vAdjustment = vChangingFactor;
  wAdjustment =wChangingFactor;

  (*These variables make sure the v, w we approximate are in the region
we where A(v,w) increases in v and w*)
  adjustedVError = vError;
  adjustedWError = wError;

  (*assurance that we are below line w =u and above line v=w*)
  If[v+2w >= VolOfS3 || v >= w,

    Print["ERROR RadiiSDB: v and w are not in increasingRegion!"];
    Print["v+ 2w = ", N[v+2w]];
    Print["Vol of S3 = ", N[VolOfS3]];
    Print["v = " ,v];
    Print["w =",w];
    ,

    While[v+adjustedVError + 2(w+adjustedWError) > VolOfS3 || \
v+adjustedVError > w+adjustedWError,

      If[v+adjustedVError > VolOfS3 - 2*w,

        adjustedVError= adjstedVError/2;

        ];

      If[2(w+adjustedWError) > VolOfS3 - (v + adjustedVError),

```

```

adjustedWError= adjustedWError/2;
adjustedVError= adjustedVError/Sqrt[2];

];

If[v+adjustedVError > w+adjustedWError,

adjustedVError = adjustedVError/2;

];

];

(*Print["adjustedVError =", adjustedVError];
Print["adjustedWError =", adjustedWError];

Print["Under w=u = ", v+adjustedVError + 2(w+
adjustedWError) < VolOfS3];

Print["While Loop test is ", (VolOneSDB[VRadius, WRadius] < v || (
VolOneSDB[VRadius, WRadius] > \
v+adjustedVError)||VolTwoSDB[VRadius, WRadius] < w || (VolTwoSDB[VRadius, \
WRadius] > w+adjustedWError))];

Print["Target V lower bound = ", N[v]];
Print["Target V upper bound = ", N[v+adjustedVError]];

Print["Target w upper bound = ", N[w]];
Print["Target w upper bound = ", N[w+adjustedWError]];

Print["Volume V at ", N[VolOneSDB[VRadius, WRadius]]];
Print["Volume W at ", N[VolTwoSDB[VRadius, WRadius]]];*)

While[(VolOneSDB[VRadius, WRadius] < v || (VolOneSDB[VRadius,
WRadius] > v+
adjustedVError)||VolTwoSDB[VRadius,
WRadius] < w || (VolTwoSDB[VRadius, WRadius] > \
w+adjustedWError)),

(*counter++;
Print["counter at ", counter];*)

(*These two variable prevent infinite while loop from occurring*)
madeChangeToV = False;
madeChangeToW = False;

If[VolOneSDB[VRadius, WRadius]< v,

VRadius = VRadius*vAdjustment;
madeChangeToV= !madeChangeToV;

];

If[VolOneSDB[VRadius, WRadius]> v+adjustedVError,

VRadius = VRadius/vAdjustment;
madeChangeToV= !madeChangeToV;

];

If[VolTwoSDB[VRadius, WRadius]< w,

WRadius = WRadius*wAdjustment;
madeChangeToW= !madeChangeToW;

];

If[VolTwoSDB[VRadius, WRadius]> w+adjustedWError,

WRadius = WRadius/wAdjustment;
madeChangeToW= !madeChangeToW;

];

If[!madeChangeToV,

```

```

vAdjustment=(vAdjustment+1)/2;
];
If[!madeChangeToW,
vAdjustment=(vAdjustment+1)/2;
];
(*Print["Volume V at ", N[VolOneSDB[VRadius, WRadius]]];
Print["Volume W at ", N[VolTwoSDB[VRadius, WRadius]]];

Print["While Loop test
is ", (VolOneSDB[VRadius, WRadius] < v || (VolOneSDB[VRadius, \
WRadius]> v+adjustedVError)||VolTwoSDB[VRadius, WRadius] <
w || (VolTwoSDB[VRadius, WRadius]>
w+adjustedWError))];*)

];
];
(*Print["It took this many steps: ", counter];*)
{VRadius, WRadius}
];

RadiiWhenWEqualsU[w_,
error_] := Module[{radius, minRadius, maxRadius, counter, adjustedError},
minRadius = 0;
maxRadius = 3Pi/2;
radius = (minRadius+maxRadius)/2;
adjustedError = error;
counter = 0;

If[3w < VolOfS3,
Print["W is too small."],
While[3(w-adjustedError) < VolOfS3,
adjustedError = adjustedError/2;
];
(*Print["Target Range = (", N[w-error], ",", N[w], ")" ];*)
While[EqualVolGivenRadius[radius] > w ||
EqualVolGivenRadius[radius] < w-adjustedError,
(*counter++;
Print["counter = ", counter];*)

If[EqualVolGivenRadius[radius] > w,
maxRadius = radius;
radius = (minRadius+radius)/2;
];
If[EqualVolGivenRadius[radius] < w-adjustedError,
minRadius = radius;
radius = (maxRadius+radius)/2;
];
(*Print["Vol = ", N[EqualVolGivenRadius[radius]]];*)
];
(*Print["It took this many steps: ", counter];*)

];
{radius, radius}
];

```

```

(*This piece of code should be used when v=w*)
RadiiWhenVEqualsW[v_, error_] := Module[{radius,
minRadius, maxRadius, counter, adjustedError},
  minRadius = 0;
  maxRadius = Pi;
  radius = (minRadius + maxRadius / 2);
  adjustedError = error;
  counter = 0;

  (*Print["radius= ", radius];*)

  If[3v >= VolOfS3,
    Print["V is too big."],

    While[3(v + adjustedError) > VolOfS3,
      adjustedError = adjustedError / 2;
    ];

    (*Print["Target Range = (", N[v-error], ",", N[v],")" ];*)

    While[EqualVolGivenRadius[
radius] < v || EqualVolGivenRadius[radius] > v + adjustedError,

      (*counter++;
      Print["counter = ", counter];*)

      If[EqualVolGivenRadius[radius] > v + adjustedError,
        maxRadius = radius;
        radius = (minRadius + radius) / 2;
      ];

      If[EqualVolGivenRadius[radius] < v,
        minRadius = radius;
        radius = (maxRadius + radius) / 2;
      ];

      (*Print["Vol = ", N[EqualVolGivenRadius[radius]]];*)

    ];

    (*Print["It took this many steps: ", counter];*)

  ];

  {radius, radius}

];

A[v_, w_, vError_,
  wError_, vChangingFactor_, \
wChangingFactor_] := Module[{radii, u, r1, r2, area},
  u = VolOfS3 - v - w;

  (*when all three volumes are equal*)
  If[v == w && w == u,
    area = 6 * Pi;

    ,
    (*one the line v=w*)
    If[v == w,
      radii = RadiiWhenVEqualsW[v, vError];
      area = AreaSDBGivenRadii[radii[[1]], radii[[2]]];

    ,
    (*on the line w=u*)
    If[w == u,

```

```

    radii = RadiiWhenWEqualsU[w, wError];
    area = AreaSDBGivenRadii[radii[[1]], radii[[2]]];

,
(*If all volumes are distinct*)

    radii = RadiiSDB[v, w, vError, wError, vChangingFactor,
    wChangingFactor];
    area = AreaSDBGivenRadii[radii[[1]], radii[[2]]];

];

];

area
];

ProofFunctionRectangle[lhs_, rhs_, p1_, p3_, left11_, right11_, left33_,
right33_, depth_] := Module[{decision, x1, y1, x2, y2, x3,
y3, left12, right12, left13, right13,
left21, right21, left22, right22, left23, right23, left31,
right31, left32, right32, d1, d2, d3, d4}, depth=1;

x1=p1[[1]];
y1=p1[[2]];

x3=p3[[1]];
y3=p3[[2]];

x2=(x1+x3)/2;
y2=(y1+y3)/2;

left12=lhs[x1,y2];right12=rhs[x1,y2];
left13=lhs[x1,y3];right13=rhs[x1,y3];
left21=lhs[x2,y1];right21=rhs[x2,y1];
left22=lhs[x2,y2];right22=rhs[x2,y2];
left23=lhs[x2,y3];right23=rhs[x2,y3];
left31=lhs[x3,y1];right31=rhs[x3,y1];
left32=lhs[x3,y2];right32=rhs[x3,y2];

(*if x1 > y3 then balancing
covers the region and we don't need to check it.*)
If[x1 > y3,

    decision =1,

    (*otherwise we do*)
    If[left11>right11&&left33>right33,

        (*then*)

        If[Min[left11, left13, left31, left33]>right33, (*then*)Print["Points \
", p1, " and ", p3, " -- Direct hit!"];
        decision=1,

        (*else*)Print["Points ", p1, " and ",
        p3, " -- Splitting into four."];

        If[ProofFunctionRectangle[lhs, rhs, {x1, y1}, {x2, y2}, left11, right11,
        left22, right22, d1] \[Equal] 1 &&
        ProofFunctionRectangle[lhs, rhs, {x2, y1}, {x3, y2}, left21, right21, \
        left32, right32, d2] \[Equal] 1 &&
        ProofFunctionRectangle[
        lhs, rhs, {x1, y2}, {x2, y3}, left12, right12, left23, right23,
        d3] \[Equal] 1 && ProofFunctionRectangle[lhs, rhs, {x2, y2}, {
        x3, y3}, left22, right22, left33, right33, d4] \[Equal] 1,

            (*then*)
            Print["Points ", p1, " and
            ", p3, " -- Hit after checking four! Depth: ", Max[depth,
            d1+1, d2+1, d3+1, d4+1]];
            decision=1,

            (*else*)Print["anyone?"];

        decision=0;
        (*endif*]);

```

```

    depth=Max[depth,d1+1,d2+1,d3+1,d4+1];
    (*endif*)Print["Oh no for ",p1," and ",p3,"!"];
    Print[N[left11]];
    Print[N[right11]];
    Print[N[left33]];
    Print[N[right33]];
    decision=0;
];

(*endif*)
];
(*return*)decision
];

ProofFunctionTriangle[lhs_,rhs_,x1_,y1_,x3_,y3_,left11_,right11_,left13_,\
right13_,left31_,right31_,depth_]:=Module[{decision,x2,y2,left12,right12,\
left21,right21,left22,right22,d1,d2,d3},depth=1;

Print["starting up the process"];

x2=(x1+x3)/2;
y2=(y1+y3)/2;
Print["whoa now!"];
If[left11>right11&&left13>right13&&left31>right31,(*then*)

If[Min[left11,
left13,left31]>right31,(*then*)Print["Points ",{x1,y1}," ",{x1,
y3}," ", and",{x3,y1}," -- Direct triangle hit!"];
decision=1,

(*else*)Print["Points ",{x1,y1}," ",{x1,y3}," ", and",{x3,y1}," --
Splitting into three."];

left12=lhs[x1,y2];right12=rhs[x1,y2];
left21=lhs[x2,y1];right21=rhs[x2,y1];
left22=lhs[x2,y2];right22=rhs[x2,y2];

If[ProofFunctionRectangle[lhs,rhs,{x1,y1},{x2,y2},left11,right11,\
left22,right22,d1][Equal]1&&ProofFunctionTriangle[lhs,rhs,x1,y2,x2,y3,left12,
right12,left13,right13,left22,right22,d2][Equal]1&&\
ProofFunctionTriangle[lhs,rhs,x2,y1,x3,y2,left21,
right21,left22,right22,left31,right31,d3][Equal]1,(*then*)
Print["Points ",{x1,y1}," ",{x1,y3}," ", and",{x3,y1}," --
Hit after checking three! Depth: ",Max[depth,d1+1,d2+1,d3+1]];
decision=1,(*else*)Print[
"Points ",{x1,y1}," ",{x1,y3}," ", and",{x3,y1}," -- Failed
after checking three!"];
decision=0;
(*endif*);
depth=Max[depth,d1+1,d2+1,d3+1];
(*endif*),(*else*)Print["Oh no for " {x1,y1},"
",{x1,y3}," ", and",{x3,y1},"!"];
Print[N[left11]];
Print[N[right11]];
Print[N[left13]];
Print[N[right13]];
Print[N[left31]];
Print[N[right31]];
decision=0;
(*endif*);
(*return*)decision
];

```

## 7.2. $H^3$ code.

```

(*error control in computation the default value of maching*)
WorkingPrecision->MachinePrecision;
COMPUTERERROR= 2^-24;
ACCURACY = 25;

%shrinkingFactorBigChanges=9999/10000;
%shrinkingFactorSmallChanges=999999/1000000;
%enlargingFactorBigChanges=101/100;
%enlargingFactorSmallChanges = 10001/10000;
%firstChange= 1/1000;
LargestSingCurvature = 16.8;
%SmallSingCurvature = 1.00509;
%SmallAdjustment = 10^(-12);

```

```

AreaSphere[k_Real]:=SetAccuracy[(4*Pi)/(-1+k^2)-COMPUTERERROR, ACCURACY];

VolSphere[k_Real]:=SetAccuracy[Pi*(-2*
  ArcCoth[k]+Sinh[2*ArcCoth[k]]), ACCURACY];

ASC1close[k1_Real,k2_Real]:=SetAccuracy[(2*Pi*(1+Sqrt[-((k1^2*(-1+Cos[(1/6)*(\
  Pi-6*ArcTan[Sqrt[3]*(k1-k2)]/(
    k1+k2))])^2)))/(
    k1^2-Cos[(1/6)*(
      Pi-6*ArcTan[(
        Sqrt[3]*(k1-
          k2))/(k1+k2)])]^2)))]/(-1+k1^2), \
  ACCURACY+1];

ASC1far[k1_Real,
  k2_Real]:=SetAccuracy[-((2*Pi*(-1+Sqrt[-((k1^2*(-1+Cos[(1/6)*(Pi-6*\
  ArcTan[Sqrt[3]*(k1-k2)]/(
    k1+k2))])^2)))/(k1^2-Cos[(
    1/6)*(Pi-6*ArcTan[(
      Sqrt[3]*(
        k1-k2))/(k1+k2)])]^2)))]/(-1+
    k1^2), ACCURACY+1];

ASC1[k1_Real,k2_Real]:=SetAccuracy[If[(k1-k2)/(k1+k2)-1/3<0,ASC1close[
  k1,k2],ASC1far[k1,k2]], ACCURACY+1];

ASC2[k1_Real,k2_Real]:=SetAccuracy[(2*Pi*(1+Sqrt[(k1^2-k2^2*Cos[(1/6)*(Pi-6*
  ArcTan[Sqrt[3]*(k1-k2)]/(k1+
    k2)])]^2)))/(
    k1^2-Cos[(1/
      6)*(Pi-6*ArcTan[(Sqrt[3]*(k1-k2)]/(k1+
        k2)])]^2)))]/(-1+k2^2), ACCURACY+1];

ASC3[k1_Real,k2_Real]:=SetAccuracy[(2*Pi*(1-Sqrt[(k1^2-(k1-k2)^2*Cos[(
  1/6)*(Pi-6*
    ArcTan[(Sqrt[3]*(
      k1-k2)]/(k1+
        k2)])]^2)/(k1^2-
          Cos[(1/6)*(
            Pi-6*ArcTan[(
              Sqrt[3]*(k1-
                k2))/(k1+k2)])]^2)))]/(-1+(k1-k2)^2), \
  ACCURACY+1];

AreaDblBubble[k1_Real,k2_Real]:=SetAccuracy[ASC1[k1,k2]+ASC2[k1,
  k2]+ASC3[k1,k2]+3*COMPUTERERROR, ACCURACY];

VolCap1close[k1_Real,k2_Real]:=-((1/(-1+k1^2))*(Pi*((-1+k1^2)*
  ArcCoth[k1]+(-1+k1^2)*ArcTanh[Sqrt[-((k1^2*(-1+Cos[(1/6)*(Pi-6*ArcTan[(\
  Sqrt[3]*(k1-k2)]/(k1+k2))])^2)))/(k1^2-Cos[(1/6)*(Pi-6*ArcTan[(Sqrt[3]*(k1-k2))\
  /((k1+k2))])^2)))]/k1-k1*(1+Sqrt[-((
    k1^2*(-1+Cos[(1/6)*(Pi-6*ArcTan[(Sqrt[\
  3]*(k1-k2)]/(k1+k2))])^2)))/(k1^2-Cos[(1/6)*(Pi-6*ArcTan[(
    Sqrt[3]*(k1-k2)]/(k1+k2))])^2))]]))));

VolCap1far[k1_Real,k2_Real]:=Pi*(-ArcCoth[k1]+(1/(-1+k1^2))*(k1+(-1+k1^2)*\
  ArcTanh[Sqrt[-((k1^2*(-1+Cos[(1/6)*(Pi-6*ArcTan[(Sqrt[
  3]*(k1-k2)]/(
    k1+k2))])^2)))/(k1^2-Cos[(1/6)*(Pi-6*ArcTan[(Sqrt[
    3]*(k1-
      k2))/(k1+
        k2)])]^2)]/
    k1-k1*
      Sqrt[-((k1^2*(-1+Cos[(1/6)*(Pi-6*\
  ArcTan[(Sqrt[3]*(k1-k2)]/(k1+k2))])^2)))/(k1^2-Cos[(
    1/6)*(Pi-6*ArcTan[(Sqrt[3]*(k1-k2)]/(k1+
      k2)])]^2)))]));

VolCap1[k1_Real,k2_Real]:=If[(k1-k2)/(k1+k2)-1/3<0,
  VolCap1close[k1,k2],VolCap1far[k1,k2]];

VolCap2[k1_Real,k2_Real]:=Pi*(-(k2/(1-k2^2))-ArcCoth[k2]-
  ArcTanh[Sqrt[(k1^2-k2^2*Cos[(1/6)*(Pi-6*ArcTan[(Sqrt[3]*(
    k1-k2)]/(k1+k2))])^2)/(k1^2-Cos[(
    1/6)*(Pi-6*
      ArcTan[(Sqrt[3]*(
        k1-k2)]/(k1+k2))])^2)]/k2)-(k2*Sqrt[(k1^2-\
  k2^2*Cos[(1/6)*(Pi-6*ArcTan[(Sqrt[3]*(k1-k2)]/(k1+k2))])^2)/(k1^2-
    Cos[(1/6)*(Pi-6*ArcTan[(Sqrt[
    3]*(k1-k2)]/(k1+k2))])^2]])]/(1-k2^2));

VolCap3[k1_Real,
  k2_Real]:=Re[Pi*(ArcTanh[Sqrt[(k1^2-(k1-k2)^2*Cos[(1/6)*(Pi-6*ArcTan[(\

```

```

Sqrt[3]*(k1-k2)/(k1+k2))]^2)/(k1^2-
Cos[(1/6)*(Pi-6*ArcTan[(Sqrt[
3]*(k1-k2))/(k1+k2)]]^2)/(k1-k2)]+(1/(-\
1+k1^2-2*k1*k2+k2^2))*(k1-k2-(-1+k1^2-2*
k1*k2+k2^2)*
ArcCoth[
k1-k2]+(-k1+
k2)*Sqrt[(k1^2-(k1-k2)^2*Cos[(1/6)*(Pi-6*ArcTan[
(Sqrt[3]*(k1-k2))/(k1+k2)]]^2)/(k1^2-Cos[(1/6)*(Pi-6*ArcTan[
Sqrt[3]*(k1-k2))/(k1+k2)]]^2)]]));
VolBub1[k1_Real,k2_Real]:=If[k1 < 1 || k2 < 1,
Print["VolBub1 Error : k1 or k2 too small"];
Print["k1 = ", k1, " k2 = ", k2 ];
,If[k1 \[Equal] k2,VolCap1[k1,k2]
,
VolCap1[k1,k2]+VolCap3[k1,k2]
]
];

VolBub2[k1_Real,k2_Real]:=If[k1\[Equal]k2, VolCap2[k1,k2]
,
VolCap2[k1,k2]-VolCap3[k1,k2]
];

VolBub1A[k1_Real, k2_Real]:=If[k1>= k2,
VolBub1[k1, k2]
,
(*Print["VolBub1A curvatures switched"];*)
VolBub2[k2, k1]
];

VolBub2A[k1_Real, k2_Real]:=If[k1>= k2,
VolBub2[k1, k2]
,
(*Print["VolBub2A curvatures switched to ", VolBub1[k1, k2]];*)
VolBub1[k2, k1]
];

VolBubV[k1_Real, k2_Real] := SetAccuracy[VolBub1A[k1, k2], ACCURACY];

VolBubW[k1_Real, k2_Real] := SetAccuracy[VolBub2A[k1, k2], 15];

(*gives an underapproximation for the curvature of a sphere in H3*)
CurvatureEstimatorSingle[v_Real] := If[v > .0001,
If[ v < .008,
8.109,
If[ v<.02,
6.,
If[
(*.02 v < 1*)
v < 1,
1.84871,
If[
(*1 v < 10*)
v<10,
1.19394,
If[
(*20 v < 30 *)
v < 30,
1.08108,
If[
(*30 v < 50*)
v<50,
1.05231,
If[
(*50 v < 60*)
v< 60,
1.04456,

```

```

If[
  (*60 <= v < 70*)
  v < 70,

  1.03883,

  If[
    (*70 v < 80*)
    v < 80,

    1.03437,

    If[
      (*80 v < 90*)
      v < 90,

      1.03086,

      If[
        (*90 v < 110*)
        v < 110,

        1.0256,

        If[
          (*110 < v < 200*)
          v < 200,
          1.01468,
          (*else 550 > v 110*)
          1.00509
        ]
      ]
    ]
  ]
],
(*else if v < .0001*)
Print["volume to small"];
(*return*)
-1
];

(*This function uses interpolation to get asure that the curvature used in \
the approximation is less than the given v within \
an interval that is small. The constant term COMPUTERERROR assures that \
the precision of the computer does not affect our lower bound*)
CurvaturefromVolSingle[v_Real, error_Real]:=Module[{
  curvature, smallK = CurvatureEstimatorSingle[v], bigK = \
LargestSingCurvature, counter =0},

  curvature = smallK;

  (**)
  While[VolSphere[curvature] > v - COMPUTERERROR || VolSphere[curvature] \
< v - error ,

  curvature = (bigK+smallK)/2;
  counter = counter +1;

  (*this tightens the interval that curvatures can be in*)
  If[VolSphere[curvature] > v - COMPUTERERROR,

  smallK= curvature;

  ,If[VolSphere[curvature] < v - error,

  bigK = curvature;

  ];
];
];
(*Print["It took ", counter, "steps." ];*)
curvature
];

```

```

(*this function returns curvature pair that is at least one box size over \
then returns the array with the new curvature pair and the changes for each \
curvature though this seems to do the same thing as CurvaturesfromVolDouble[] \
this doesn't limit the size of the change*)
MakeChangeOneBoxSizeOver[k1_Real, k2_Real, volume1_Real, BoxSize_Real, \
change1_Real] := Module[{minVolChange, maxVolChange, adjCurvature1, \
adjCurvature2, adjChange1, finalArray, counter, slope, closestOverShot, \
closestUnderShot, lastCurvatureGuess},

  minVolChange = .8*BoxSize;
  maxVolChange = 1.7*BoxSize;

  adjCurvature1 = change1*k1;
  adjCurvature2 = k2;
  adjChange1 = change1;
  counter=0;

  closestOverShot= k2;
  closestUnderShot = k1;

  slope = (VolBub1A[adjCurvature1, adjCurvature2] - \
volume1)/(adjCurvature1 - k1);

  adjCurvature1 = adjCurvature1 + BoxSize/slope;

  (*Print["curvatures, k1 = ", adjCurvature1, " k2 is ", k2];*)

  While[ (VolBub1A[adjCurvature1, adjCurvature2]- volume1 < minVolChange || \
VolBub1A[adjCurvature1, adjCurvature2]- volume1 > maxVolChange) && \
adjCurvature1 > 1 && counter < 10,

    If[VolBub1A[adjCurvature1, adjCurvature2]- volume1 < minVolChange,

      (*Print["hi test 1 is ", VolBub1A[
adjCurvature1, adjCurvature2]- volume1 < minVolChange];*)

      lastCurvatureGuess = adjCurvature1;

      adjCurvature1 = (adjCurvature1 + closestOverShot)/2;

      closestOverShot = lastCurvatureGuess;
    ];

    If[VolBub1A[adjCurvature1, adjCurvature2]- volume1 > maxVolChange,

      (*Print["hi test 2 is ", VolBub1A[adjCurvature1, adjCurvature2]-
volume1 > maxVolChange];*)

      lastCurvatureGuess = adjCurvature1;

      adjCurvature1 = (adjCurvature1 + closestUnderShot)/2;

      closestUnderShot = lastCurvatureGuess;
    ];

    (*Print["trial", counter, " vol is \
", VolBub[adjCurvature1, adjCurvature2]];*)

    counter= counter+1;
    (*Print["counter = ", counter];
Print["vol1 = ",
VolBub1A[adjCurvature1, adjCurvature2] ];
Print["change = \
", adjChange1];
Print["curvature ", adjCurvature1 ];
*)

  ];

  If[adjCurvature1 <= 1,

    Print["MakeChangeOneBoxSizeOver error curvature too small"];
  ];

  (*
Print["vol1 = ", VolBub1[adjCurvature1, adjCurvature2] ];
Print["change = ", adjChange1];
Print["curvature ", adjCurvature1];
*)
  adjChange1 = adjCurvature1/k1;

```

```

{adjCurvature1, adjChange1}
];

(*this function returns curvature pair that is at least one box size over \
then returns the array with the new curvature pair and the changes for each \
curvature though this seems to do the same thing as CurvaturesfromVolDouble[] \
this doesn't limit the size of the change*)
MakeChangeOneBoxSizeUp[k1_Real, k2_Real, volume2_Real, BoxSize_Real, \
change2_Real] := Module[{minVolChange, maxVolChange, adjCurvature1, \
adjCurvature2, adjChange1, counter, slope, lastCurvatureGuess, \
closestUnderShot, closestOverShot},

minVolChange = 1.*BoxSize;
maxVolChange = 2.*BoxSize;

adjCurvature1 = k1;
adjCurvature2 = k2*change2;
adjChange2 = change2;
closestUnderShot=k2;
closestOverShot = 1.;
(*Might want to change this!*)

counter=0;

(*Print["hi"];
Print["test =", (VolBub2[adjCurvature1,adjCurvature2]- volume2< \
minVolChange || VolBub2[adjCurvature1,
adjCurvature2]- volume2 > maxVolChange) &&
adjCurvature2 > 1 && counter < 10];
*)

slope = (
VolBub2A[adjCurvature1, adjCurvature2] - volume2)/(adjCurvature2 - k2);

adjCurvature2 = adjCurvature2 + BoxSize/slope;

While[ (VolBub2A[adjCurvature1,
adjCurvature2]- volume2< minVolChange || \
VolBub2A[adjCurvature1,adjCurvature2]-
volume2 > maxVolChange) && adjCurvature2 > 1 && counter < 10,

counter= counter+1;
(*Print["counter = ", counter];
Print["vol2 =", VolBub2[adjCurvature1,adjCurvature2] ];
Print["change = ", adjChange2];
Print["curvature ", adjCurvature1 ];
*)

If[VolBub2A[adjCurvature1,adjCurvature2]- volume1< minVolChange,

lastCurvatureGuess = adjCurvature2;

adjCurvature2 = (adjCurvature2 + closestOverShot)/2;

closestOverShot = lastCurvatureGuess;
];

If[VolBub2A[adjCurvature1,adjCurvature2]- volume1> maxVolChange,

lastCurvatureGuess = adjCurvature1;

adjCurvature2 = (adjCurvature2 + closestUnderShot)/2;

closestUnderShot = lastCurvatureGuess;
];

];

(*Print["counter = ", counter];*)

If[adjCurvature2 <= 1,

Print["MakeChangeOneBoxSizeUp error curvature too small"];
];

(*
Print["vol2 =", VolBub2[adjCurvature1,adjCurvature2] ];
Print["change = ", adjChange2];
Print["curvature ", adjCurvature2];
*)

```

```

adjChange2 = adjCurvature2/k2;

{adjCurvature2, adjChange2}
];

(*This function tells whether the point is in given rectangle with lower left \
corner xValueBox, yValueBox and width boxWidth and height boxHeight*)
IsPointInBox[xValuePoint_Real, yValuePoint_Real, xValueBox_Real, \
yValueBox_Real, boxWidth_Real, boxHeight_Real] :=
(xValuePoint > xValueBox + COMPUTERERROR && xValuePoint < (xValueBox + \
boxWidth) && yValuePoint > yValueBox + COMPUTERERROR &&
yValuePoint < (yValueBox + boxHeight));

(*This function tells whether the point has a x value and a y value bigger than \
a given x, and y value *)
IsPointToUpperRight[xValuePoint_Real, yValuePoint_Real, xValueBox_Real, \
yValueBox_Real] :=
(xValuePoint > xValueBox) && yValuePoint > yValueBox ;

(*This function is supposed to manipulate a given curvature pair for volumes \
vol1 and vol2 and get a curvature pair corresponding to volumes inside a \
box with v1, v2 as its lower left corner*)
(*right now this method needs to dynamically choose its changing size to avoid \
infinite loops*)
(*Returns a two element list of curvatures*)
(*v1, v2 should correspond to the point in the lower right of the rectangle \
containing VolBub1A[k1, k2], VolBub2A[k1, k2]
the curvatures k1, k2 should be a point in the middle of the rectangle*)

NextCurvaturePairinArray[ k1_Real, k2_Real, v1_Real, v2_Real, change1_Real, \
change2_Real, scalingFactor1_Real, scalingFactor2_Real] \
:=Module[{curvatureComponent11, curvatureComponent12, curvatureComponent21, \
curvatureComponent22, startingVolume1, startingVolume2, volumeComponent11, \
volumeComponent12, volumeComponent21, volumeComponent22, alpha, beta, \
volumeOneTarget, volumeTwoTarget, curvaturePair, volumeChangeMatrix, \
startingVoltoTargetMatrix, alphaBetaMatrix, counter, \
oneBoxSizeOverArray, oneBoxSizeUpArray},

(*adjustment arrays*)
oneBoxSizeOverArray =
MakeChangeOneBoxSizeOver[k1, k2,
VolBub1A[k1,k2], change1, scalingFactor1];

oneBoxSizeUpArray = MakeChangeOneBoxSizeUp[k1, k2, VolBub2A[k1, k2], \
change2, scalingFactor2];

(*Declaration of initial curvaturevectors*)

curvatureComponent11 = oneBoxSizeOverArray[[1]];

curvatureComponent12 = k2;
curvatureComponent21 = k1;
curvatureComponent22 = oneBoxSizeUpArray[[1]];
counter=0;

(*Print["c11 = ", curvatureComponent11];
Print["c12 = ", curvatureComponent12];
Print["c21 = ", curvatureComponent21];
Print["c22 = ", curvatureComponent22];*)

(*declaration of initial volumevectors*)
startingVolume1 = VolBub1A[k1, k2];
startingVolume2 = VolBub2A[k1, k2];
volumeComponent11 = VolBub1A[
curvatureComponent11, curvatureComponent12];
volumeComponent12 = VolBub2A[curvatureComponent11, \
curvatureComponent12];
volumeComponent21 = VolBub1A[
curvatureComponent21, curvatureComponent22];
volumeComponent22 = VolBub2A[curvatureComponent21, \
curvatureComponent22];

(*Print["v11 = ", volumeComponent11];
Print["v12 = ", volumeComponent12];
Print["v21 = ", volumeComponent21];
Print["v22 = ", volumeComponent22];*)

```

```

(*declaration of target Volumes*)
volumeOneTarget = v1 + .5*change1;
volumeTwoTarget = v2 + .5*change2;

(*Print["volumeOneTarget = ", volumeOneTarget];
Print["volumeTwoTarget = ", volumeTwoTarget];
*)

(*declaration of Matricies*)
startingVoltoTargetMatrix= {volumeOneTarget -
startingVolume1, {volumeTwoTarget - startingVolume2}};

(*Print["first test =
",IsPointinBox[volumeComponent11, volumeComponent12, v1+change1, \
v2, change1, change2]];
*)
If[IsPointinBox[volumeComponent11, volumeComponent12, v1, v2, change1,
change2],

(*Print["v11 = ", volumeComponent11];
Print["v12 = ", volumeComponent12];
Print["change1 = ", change1];
Print["change2 = ", change2];
*)

curvaturePair = {curvatureComponent11, curvatureComponent12},

(*Print["2nd test = ",IsPointinBox[volumeComponent21, \
volumeComponent22, v1+change1, v2, change1, change2]];

Print["v21 = ", volumeComponent21];
Print["v22 = ", volumeComponent22];
Print["change1 = ", change1];
Print["change2 = ", change2];
*)

If[IsPointinBox[volumeComponent21, volumeComponent22, v1, v2, change1,
change2],

(*Print["case II"];
Print["v21 = ", volumeComponent21];
Print["v22 = ", volumeComponent22];
Print["change1 = ", change1];
Print["change2 = ", change2];*)

curvaturePair = {curvatureComponent21, curvatureComponent22},

(*Print["while loop test", \
!IsPointinBox[volumeComponent11,volumeComponent12,v1+change1, v2, change1,
change2]];
*)

volumeChangeMatrix = {{volumeComponent11 - startingVolume1, \
volumeComponent12 - startingVolume2}, {volumeComponent21 -
startingVolume1, volumeComponent22 - startingVolume2}};

(*Print["volumeChangeMatrix = ",volumeChangeMatrix//MatrixForm];
*)

alphaBetaMatrix = Inverse[volumeChangeMatrix] .
startingVoltoTargetMatrix;
(*Print["alphaBetaMatrix = ", alphaBetaMatrix//MatrixForm];
*)

alpha = alphaBetaMatrix[[1,1]];
beta = alphaBetaMatrix[[2,1]];

curvatureComponent11 = (curvatureComponent11-k1)*alpha + \
(curvatureComponent21-k1)*beta +k1;
curvatureComponent12= (curvatureComponent12-k2)*
alpha + (curvatureComponent22-k2)*beta + k2;

(*Print["inside while loop"];
Print["c11 = ", curvatureComponent11];
Print["c12 = ", curvatureComponent12];
Print["alpha = ", alpha];
Print["beta = ", beta];*)

volumeComponent11 = VolBub1A[
curvatureComponent11, curvatureComponent12];
volumeComponent12 = VolBub2A[

```

```

curvatureComponent11, curvatureComponent12];

(*Print["v11 = ", volumeComponent11];
Print["v12 = ", volumeComponent12];*)

];

curvaturePair = {curvatureComponent11, curvatureComponent12};
(*Print["after loop"];
Print["v11 = ", volumeComponent11];
Print["v12 = ", volumeComponent12];
Print["change1 = ", change1];
Print["change2 = ", change2];

Print["2nd
test = ",IsPointinBox[volumeComponent11, \
volumeComponent12, v1+change1, v2, change1, change2]];*)

];

(*If[!
IsPointinBox[VolBub1[curvatureComponent11, curvatureComponent12], \
VolBub2[curvatureComponent11, curvatureComponent12], v1+change1, v2, \
change1, change2],

curvaturePair = {-1, -1};
];
*)
curvaturePair
];

(*This returns a lower bound on the positive part of the Hutchings function \
for a given volume pair*)
LowerBoundOnPosHutchingFunction[vol1_Real, vol2_Real, error1_Real, \
error2_Real]:=

2*AreaSphere[CurvaturefromVolSingle[vol1/2, error1]]+
AreaSphere[CurvaturefromVolSingle[vol2, error2]]+
AreaSphere[CurvaturefromVolSingle[vol1+vol2, error2]];

(*This returns an array of areas is indexed by its volume pair *)
ArrayBuilderForSingAreasandCurvatures[vMin_Real, vMax_Real, boxSize_Real, \
error_Real]:=Module[
{AreaArray, volumeArray, n, g},

g[i_Integer]= AreaSphere[CurvaturefromVolSingle[vMin+ i*boxSize, \
error]];

n[i_Integer]=VolSphere[CurvaturefromVolSingle[vMin+ i*boxSize, error]];

AreaArray= Array[g, Ceiling[(vMax-vMin)/boxSize], 0];
volumeArray=Array[n, Ceiling[(vMax-vMin)/boxSize], 0];

{volumeArray,AreaArray}
];

(*This returns an array of areas is indexed by its volume pair *)
ArrayBuilderForSingAreasandVolumes[vMin_Real, vMax_Real, boxSize_Real, \
error_Real]:=Module[
{AreaArray, volumeArray, n, g},

g[i_Integer]= AreaSphere[CurvaturefromVolSingle[vMin+i*boxSize,
error]];
n[i_Integer]=vMin+i*boxSize;

AreaArray= Array[g, Ceiling[(vMax-vMin)/boxSize], 0];
volumeArray=Array[n, Ceiling[(vMax-vMin)/boxSize], 0];

{volumeArray,AreaArray}
];

(*This returns an array of areas is indexed by its volume pair *)
ArrayBuilderForSingAreas[vMin_Real, vMax_Real, \
boxSize_Real, error_Real]:=Module[
{AreaArray, g},

```

```

g[i_Integer]= AreaSphere[CurvaturefromVolSingle[vMin+i*boxSize, \
error]];

AreaArray= Array[g, Ceiling[(vMax-vMin)/boxSize], 0];

AreaArray
];

(*This code fills an array with curvature pairs. It moves on to the next \
region after verifying that 2 the area of the double bubble enclosing volumes \
v,w is less than the concave part of the Hutchings function for v,w*)
ArrayFillingProof[vMin_Real, wMin_Real, wMax_Real, rectangleHeight_Real, \
rectangleWidth_Real, startingCurvature1_Real, startingCurvature2_Real, \
adjustmentMainVol_Real, adjustmentSecondVol_Real] :=Module[{smallerVolume, \
largerVolume, nextRowStartingCurvatures, curvaturePair, failSafe, \
nextRowStartingPosition, counter, insideCounter, failingVolumeV, \
singAreasArray, singAreasArrayStart},

failSafe=True;
counter=0;
insideCounter =1;

singAreasArrayStart=(vMin-rectangleWidth)/2;
singAreasArray= ArrayBuilderForSingAreas[(vMin-rectangleWidth)/2, \
2*(wMax+rectangleWidth), rectangleWidth/2, rectangleWidth/2];
Print["Array completed, evaluating Hutchings Function"];
(*Print["singAreasArray =", singAreasArray];*)

(*Print["hi"];
Print[failSafe];*)

(*nextRowStartingPosition keeps track of the x position that \
starts a new row
initially it is set to 0 and then set to correct position*)
nextRowStartingPosition = 0.0;
nextRowStartingCurvatures = curvaturePair;

curvaturePair = {startingCurvature1, startingCurvature2};

smallerVolume = vMin;
largerVolume=wMin+rectangleHeight;

(*Print["first test", largerVolume wMax +rectangleHeight && \
failSafe];*)

While[largerVolume wMax +rectangleHeight && failSafe,

(*While the v1 is less than v2*)
While[smallerVolume largerVolume +rectangleWidth && failSafe ,

failSafe = IsPointinBox[VolBubV[curvaturePair[[
1]], curvaturePair[[2]]],
VolBubW[curvaturePair[[1]], curvaturePair[[2]]], smallerVolume, \
largerVolume, 2*rectangleWidth, 2*rectangleHeight];

If[!failSafe,

Print["( ",
VolBubV[curvaturePair[[1]], curvaturePair[[2]]], ", ",
VolBubW[curvaturePair[[1]],
curvaturePair[[2]]], ") is not in the
box defined by (" , smallerVolume, ", ", largerVolume, ") \
and (" , rectangleWidth, ", ", rectangleHeight, ") " ];

];

(*marker*)
(*this sets up moving one row up*)
If[ nextRowStartingPosition \[Equal]0 && .85 * largerVolume \
smallerVolume+rectangleWidth,
nextRowStartingPosition = smallerVolume;
nextRowStartingCurvatures = curvaturePair;

];

(*Print["hi! 11"];*)
(*If[insideCounter <= 5,

```

```

Print["target volume pair = {" , smallerVolume, " , " , largerVolume,
"}"];
Print["actual volume

pair = {" , \
VolBubV[curvaturePair[[1]],curvaturePair[[2]] ] , " , " , \
VolBubW[curvaturePair[[1]],curvaturePair[[2]] ] , ""];
Print["curvatures are " , curvaturePair];
insideCounter++;
Print["failSafe " , failSafe];

];*)

(*if there is a problem with the curvature function or the \
hutchings function is negative then the program should fail*)

(*Print["v/2 from array =" , \
singAreasArray[Floor[(smallerVolume-rectangleWidth)/2-singAreasArrayStart]/(\
rectangleWidth/2)+1]]];
Print["index =" , \
Floor[(smallerVolume-rectangleWidth)/2-singAreasArrayStart]/(rectangleWidth/\
2)+1]];

Print["v/2 not from array =" , \
AreaSphere[CurvaturefromVolSingle[(\
smallerVolume-rectangleWidth)/2 , \
rectangleWidth/2]];

Print["w from array =" , \
singAreasArray[Floor[
2+(largerVolume-rectangleWidth-singAreasArrayStart)/(\
rectangleWidth/2)]]];

Print["index =" , \
Floor[2+(largerVolume-rectangleWidth-singAreasArrayStart)/(rectangleWidth/2)]]\
;

Print["w not from array =" \
,AreaSphere[CurvaturefromVolSingle[largerVolume-
rectangleWidth , rectangleWidth/2]];

Print["v+w from array =" , \
singAreasArray[Floor[1+(smallerVolume+largerVolume-rectangleWidth-
singAreasArrayStart)/(rectangleWidth/2)]]];

Print["index =" , \
Floor[1+(-rectangleWidth+smallerVolume+largerVolume-singAreasArrayStart)/(\
rectangleWidth/2)]];

Print[
"v+w not from array =" \
,AreaSphere[CurvaturefromVolSingle[smallerVolume+largerVolume-rectangleWidth , \
rectangleWidth/2]];

Print["v + w =" , smallerVolume+largerVolume-rectangleWidth];
Print["v =" , smallerVolume];
Print["w =" , largerVolume-rectangleWidth];

*)

If[curvaturePair[[1]] < 1 \
||2*singAreasArray[Floor[(smallerVolume-rectangleWidth)/2-\
singAreasArrayStart]/(rectangleWidth/2)+1]]+singAreasArray[[
Floor[2+(largerVolume-rectangleWidth-\
singAreasArrayStart)/(rectangleWidth/\
2)]]]+singAreasArray[Floor[1+(smallerVolume+\
largerVolume-rectangleWidth-singAreasArrayStart)/(rectangleWidth/2)]]+ - \
2*AreaDblBubble[curvaturePair[[1]] , curvaturePair[[2]]] < 0,

Print["curvaturePair =" , curvaturePair];
Print["hutchings function =" , \
2*singAreasArray[Floor[(smallerVolume-rectangleWidth)/2-
singAreasArrayStart]/(rectangleWidth/2)+1]]+singAreasArray[[\
Floor[2+(largerVolume-rectangleWidth-singAreasArrayStart)/(rectangleWidth/2)]]\
]+singAreasArray[Floor[1+(smallerVolume+largerVolume-rectangleWidth-\
singAreasArrayStart)/(rectangleWidth/2)]]] - 2*AreaDblBubble[
curvaturePair[[1]] , curvaturePair[[2]]];
failSafe = False;

```

```

failVolumeV =smallerVolume;

Print["upper
bound on dbl bubble", 2*AreaDblBubble [curvaturePair[[1]], \
curvaturePair[[2]]];
Print["lower bound on concave",singAreasArray[[
Floor[((smallerVolume-rectangleWidth)/2-\
singAreasArrayStart)/(rectangleWidth/2)+1]]+singAreasArray[[Floor[2+(\
largerVolume-
rectangleWidth-singAreasArrayStart)/(rectangleWidth/2)]]]+\
singAreasArray[[Floor[1+(smallerVolume+largerVolume-rectangleWidth-
singAreasArrayStart)/(rectangleWidth/2)]]]];

];

(*Print["inside while loop"];*)
curvaturePair = NextCurvaturePairinArray[curvaturePair[[1]], \
curvaturePair[[2]], smallerVolume+rectangleWidth, largerVolume,
rectangleWidth, rectangleHeight, adjustmentMainVol, \
adjustmentSecondVol];

(*Print["upper bound on dbl bubble", 2*AreaDblBubble \
[curvaturePair[[1]], curvaturePair[[2]]]];
Print["lower bound on \
concave",LowerBoundOnPosHutchingFunction[smallerVolume, largerVolume, \
rectangleWidth, rectangleHeight]];
*)

smallerVolume = smallerVolume+ rectangleWidth;

];

If[failSafe,
(*Print["smallerVolume", smallerVolume];*)
smallerVolume = nextRowStartingPosition;

nextRowStartingPosition =0.0;
counter++;
insideCounter= 1;

If[counter \[Equal]100,
Print["w at ", largerVolume];
counter=0;
];

(*Print["SETTING the curvature pair"];
Print["Volumes before {" , \
VolBubV[nextRowStartingCurvatures[[1]],nextRowStartingCurvatures[[2]]], " , \
",VolBubW[nextRowStartingCurvatures[[1]],
nextRowStartingCurvatures[[2]]], "}"];

Print["target box { (" , smallerVolume ",", \
smallerVolume+rectangleHeight , ") X (
", (largerVolume+rectangleHeight), ", ", \
(largerVolume+2*rectangleHeight)" }"];
*)
largerVolume = largerVolume+ rectangleHeight;

While[!IsPointinBox[VolBubV[curvaturePair[[1]],
curvaturePair[[2]]], VolBubW[curvaturePair[[1]],curvaturePair[[
2]]], smallerVolume, largerVolume, 2*rectangleWidth, \
2*rectangleHeight],
curvaturePair = NextCurvaturePairinArray[curvaturePair[[1]], \
curvaturePair[[2]], smallerVolume, largerVolume+rectangleHeight, \
rectangleWidth, rectangleHeight, adjustmentMainVol, adjustmentSecondVol];
];
(*Print["VOL1 =",VolBubV[curvaturePair[[1]], curvaturePair[[2]]] ];
Print["VOL2 =",VolBubW[curvaturePair[[1]], curvaturePair[[2]]] ];
*)

failSafe=IsPointinBox[VolBubV[curvaturePair[[1]],curvaturePair[[2]]]\
, VolBubW[curvaturePair[[1]],curvaturePair[[2]]], smallerVolume, largerVolume, \
2*rectangleWidth, 2*rectangleHeight]&&smallerVolume/largerVolume.85;

If[!failSafe,

```

```

Print["error at end ( ",VolBubV[
  curvaturePair[[1]], curvaturePair[[2]]], ",
  ", VolBubW[curvaturePair[[1]],curvaturePair[[2]]], ")
  is not in the box defined
  by (", smallerVolume," ", largerVolume,")
  and (", rectangleWidth, " ", rectangleHeight, ")" ];

];

(*Print["test ",IsPointinBox[VolBubV[curvaturePair[[1]],
curvaturePair[[
2]]], VolBubW[curvaturePair[[1]],curvaturePair[[2]]], smallerVolume,
largerVolume, rectangleWidth, rectangleHeight]];*)
];

];

(*Print["while loop test ", smallerVolume largerVolume && failSafe, \
failSafe];
Print["first part of test", smallerVolume largerVolume;
*)

If[failSafe,
Print["done"],

Print["failed at (",smallerVolume, " ", largerVolume, ")"];
Print["ratio =",smallerVolume/largerVolume];
];

Print["failSafe = ", failSafe];
];

```

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