Find out if the following series converge or diverge. Give reasons to support your findings.

$$\sum_{n=1}^{\infty} \frac{e^n + 5^{n-2}}{7^n}$$

We would like to say:

$$\sum_{n=1}^{\infty} \frac{e^n + 5^{n-2}}{7^n} = \sum_{n=1}^{\infty} \frac{e^n}{7^n} + \sum_{n=1}^{\infty} \frac{5^n \cdot 5^{-2}}{7^n}$$

but we can only do that if both $\sum_{n=1}^{\infty} \frac{e^n}{7^n}$ and $\sum_{n=1}^{\infty} \frac{5^n \cdot 5^{-2}}{7^n}$ converge.
Notice that $\sum_{n=1}^{\infty} \frac{e^n}{7^n}$ is a geometric series with $r = \frac{e}{7} < 1$ and $\sum_{n=1}^{\infty} \frac{1}{25} \frac{5^n}{7^n}$ is also a geometric series with $r = \frac{5}{7} < 1$, so both series converge.

Since both $\sum_{n=1}^{\infty} \frac{e^n}{7^n}$ and $\sum_{n=1}^{\infty} \frac{5^n \cdot 5^{-2}}{7^n}$ converge, then,

$$\sum_{n=1}^{\infty} \frac{e^n + 5^{n-2}}{7^n} = \sum_{n=1}^{\infty} \frac{e^n}{7^n} + \sum_{n=1}^{\infty} \frac{5^n \cdot 5^{-2}}{7^n}$$

and so $\sum_{n=1}^{\infty} \frac{e^n + 5^{n-2}}{7^n}$ converges.

$$\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{-3n}$$

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-3n} = \lim_{n\to\infty} \left(\left(1 + \frac{1}{n}\right)^{n}\right)^{-3} = e^{-3} \neq 0$$

Since $\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-3n} \neq 0$, $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{-3n}$ diverges by the test for divergence.

$$\sum_{n=1}^{\infty} (2n)^{\frac{2}{n}}$$

$$\lim_{n\to\infty} (2n)^{\frac{2}{n}} = \lim_{n\to\infty} 2^{\frac{2}{n}} (n^{\frac{1}{n}})^2 = 1 \cdot 1^2 = 1 \neq 0$$

Since $\lim_{n\to\infty}(2n)^{\frac{2}{n}} \neq 0$, $\sum_{n=1}^{\infty}(2n)^{\frac{2}{n}}$ diverges by the test for divergence.

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3+5}$$

(Notice that the difference in powers is 1 so our series should look like $\frac{1}{n}$. We want to find a test that will capture that idea. The limit comparison test will do just that.)

$$\lim_{n\to\infty} \frac{\frac{n^2}{n^3+5}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^3+5} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^3}{n^3+5} = 1$$

Since $\lim_{n\to\infty} \frac{\frac{n^2}{n^3+5}}{\frac{1}{n}} = 1$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \frac{n^2}{n^3+5}$ diverges by the limit comparison test.

$$\sum_{n=1}^{\infty} \frac{1}{n+\ln n}$$

(You might first think, hey here is a series with only n's and $\ln n$'s I should use the K_{pq} test. Well that's a good idea but remember the K_{pq} test only deals with things being multiplied together, so the fact that n and $\ln n$ are being added causes problems.)

(Instead let's use the idea that $\ln n$ is insignificant when compared to n. So in the limit, the terms $\frac{1}{n+\ln n}$ should behave like $\frac{1}{n}$).

$$\lim_{n\to\infty} \frac{\frac{1}{n+\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n+\ln n} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n}{n+\ln n} = \lim_{n\to\infty} \frac{1}{1+\frac{\ln n}{n}} = 1$$

since $\lim_{n\to\infty} \frac{\ln n}{n} = 0$.

Since $\lim_{n\to\infty} \frac{\frac{1}{n+\ln n}}{\frac{1}{n}} = 1$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (by the p-test), $\sum_{n=1}^{\infty} \frac{1}{n+\ln n}$ diverges by the limit comparison test.

(Incidently, I am sorry not to have chosen a series where the limit that comes out of the limit comparison test was not 1.)

$$\sum_{n=1}^{\infty} \frac{3}{n!}$$

(This is an example of a series that converges ridiculously fast. If this were Spaceballs, this series would converge at plaid speed. So fast that the limit comparison test will have trouble capturing that idea, the convergence of the series happens quickly and in some sense this can't be seen by the limit. Instead, we will use the comparison test.)

$$\sum_{n=1}^{\infty} \frac{3}{n!} = \frac{3}{1!} + \sum_{n=2}^{\infty} \frac{3}{n!} = 3 + \sum_{n=2}^{\infty} \frac{3}{n!}$$

If $n \geq 2$, $\frac{3}{n!} = \frac{3}{n(n-1)(n-2)!}$ and $\frac{3}{n(n-1)(n-2)!} \leq \frac{3}{n(n-1)}$. We can see this if we cross multiply we get $3n(n-1) \leq 3n(n-1)(n-2)!$. This is true since $(n-2)! \geq 1$ if $n \geq 2$.

Now we want to find out more about $\sum_{n=1}^{\infty} \frac{3}{n(n-1)}$

Notice that,

$$\lim_{n\to\infty} \frac{\frac{3}{n(n-1)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{3}{n^2 - n)} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{3n^2}{n^2 - n)} = 3.$$

Since $\lim_{n\to\infty} \frac{\frac{3}{n(n-1)}}{\frac{1}{n^2}} = 3$, $0 < 3 < \infty$, and $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges by the p-test, $\sum_{n=1}^{\infty} \frac{3}{n(n-1)}$ converges by the limit comparison test.

Now, since $\frac{3}{n(n-1)(n-2)!} \leq \frac{3}{n(n-1)}$ and $\sum_{n=1}^{\infty} \frac{3}{n(n-1)}$ converges, $\sum_{n=1}^{\infty} \frac{3}{n!}$ converges by the comparison test.

$$\sum_{n=1}^{\infty} \frac{2^n + n^2}{3^n}$$

$$\lim_{n\to\infty} \frac{\frac{2^n+n^2}{3^n}}{\frac{2^n}{3^n}} = \lim_{n\to\infty} \frac{2^n + n^2}{3^n} \cdot \frac{3^n}{2^n} = \lim_{n\to\infty} \frac{1 + \frac{n^2}{2^n}}{1} \cdot \frac{3^n}{3^n} = 1$$

since $\lim_{n\to\infty} \frac{n^2}{2^n} = 0$.

Since $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$ converges by the geometric series test ($\frac{2}{3} < 1$) and $\lim_{n\to\infty} \frac{\frac{2^n+n^2}{3^n}}{\frac{2^n}{3^n}} = 1$, $\sum_{n=1}^{\infty} \frac{2^n+n^2}{3^n}$ converges by the limit comparison test.

$$\sum_{n=1}^{\infty} \frac{3^n + n^2}{2^n}$$

(You could use the limit comparison test here and your argument would be similar to the previous argument except the answer would come out that or series diverges since $\sum_{n=1}^{\infty} \frac{3^n}{2^n}$ diverges. I am going to use the comparison test, because it is easier in this case.)

Notice $\frac{3^n+n^2}{2^n} \geq \frac{3^n}{2^n}$ since the denominators are the same but the numerator on the left is bigger. Another way to say this is, $\frac{3^n+n^2}{2^n} \geq \frac{3^n}{2^n}$ since $n^2 \geq 0$.

$\sum_{n=1}^{\infty} \frac{3^n}{2^n}$ diverges by the geometric series test since $\frac{3}{2} > 1$.

Since $\frac{3^n+n^2}{2^n} \geq \frac{3^n}{2^n}$ and $\sum_{n=1}^{\infty} \frac{3^n}{2^n}$ diverges, $\sum_{n=1}^{\infty} \frac{3^n+n^2}{2^n}$ by the comparison test.

Things to keep in mind

Notice the style of how these problems were solved. "Give a reason" means construct an argument. The last line of every problem is a sentence that justifies the convergence or divergence of the series, where every part of the tests are explained/acknowledged. When using a comparison test, be especially careful about knowing whether the series you choose to compare things too converges or diverges. In the third and forth examples notice, I took the time to show that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Your work on quizzes and exams should also have this component. It is good practice to do your homework problems this way as well.