Algebra Review Problem Solutions

1. $$\frac{(-6)(-2)}{(-4)} = \frac{12}{-4} \left(= \frac{-12}{4} = -\frac{12}{4} \right) = \boxed{-3}.$$

Common mistake: Thinking that the negative sign on the bottom cancels *both* of the negative signs on top, and getting 3.

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2. $$\frac{2a + 10a^2}{2a} = \frac{2a(1+5a)}{2a} = \boxed{1+5a}.$$

Common mistake: Thinking that the $2a$ in the denominator (bottom) just obliterates the $2a$ in the numerator (top) without affecting the $10a^2$, and getting either $1 + 10a^2$ or $10a^2$.

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3. $\sqrt{s^2+t^2}$: $\boxed{\text{Already simplified.}}$

Common mistake: "Distributing" the square root over the addition (in violation of commandment #14), getting

$$\sqrt{s^2+t^2} = \sqrt{s^2} + \sqrt{t^2} = s + t.$$

(Actually, it's also untrue that $\sqrt{s^2} = s$; in fact, $\sqrt{s^2} = |s|$.)

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4. $$5(y^5 - y - 2) - (3y^2 - 7y + 1) = 5y^5 - 5y - 10 - 3y^2 + 7y - 1 = \boxed{5y^5 - 3y^2 + 2y - 11}.$$

Common mistake: Not distributing correctly. Most people remember to distribute the 5 in the first term, but lots of folks will forget to distribute the minus sign in the second term. It's exactly like you're multiplying by -1.

Incorrect: $-(3y^2 - 7y + 1) = -3y^2 - 7y + 1$

Correct:
$$\begin{aligned} -(3y^2 - 7y + 1) &= (-1)\cdot(3y^2 - 7y + 1) \\ &= (-1)\cdot(3y^2) + (-1)(-7y) + (-1)\cdot 1 = -3y^2 + 7y - 1. \end{aligned}$$

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5. $$\frac{x^2-4}{x-2} = \frac{(x+2)(x-2)}{x-2} = \boxed{x+2}.$$

Common mistake: Some sort of crazy cancellation (commandment #5). For instance, someone might think that since $x^2/x = x$, and $4/2 = 2$, it must be $x - 2$, or something. This is, of course, witchcraft. The only way you can perform a cancellation of the kind we would like here, is if you can factor something out of the *entire* numerator and also factor the same thing out of the *entire* denominator. In this case, you can: $x - 2$ factors cleanly out of both top and bottom. This is easy to spot because $x^2 - 4$ is a difference of two squares. (Remember the general fact that $a^2 - b^2 = (a+b)(a-b)$.)

Actually, there's a small lie here: $\frac{x^2-4}{x-2}$ and $x+2$ are not exactly the same thing. If $x \neq -2$, then they are exactly the same. But if $x = -2$, then $x+2$ is defined $(-2+2=0)$, but $\frac{x^2-4}{x-2}$ isn't—you get $0/0$, which is meaningless (commandment #1). So to be completely accurate, we could say

$$\frac{x^2-4}{x-2} = \begin{cases} x+2, & \text{if } x \neq -2, \\ \text{undefined}, & \text{if } x = -2. \end{cases}$$

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6. $\sqrt{12} + \sqrt{27} = \sqrt{2^2 \cdot 3} + \sqrt{3^2 \cdot 3} = 2\sqrt{3} + 3\sqrt{3} = (2+3)\sqrt{3} = \boxed{5\sqrt{3}}.$

Common mistake: To think that $\sqrt{12} + \sqrt{27} = \sqrt{12+27} = \sqrt{39}$. This is untrue. (Check it on a calculator.) This is similar in flavor to the mistake many people make on problem #3, thinking that square roots "distribute" over addition. (See commandment #14.)

Incidentally, the fact that $\sqrt{12}+\sqrt{27}$ does really simplify down to one fairly simple expression here is kind of a coincidence. Both 12 and 27 have the form "square times three," and that's why it came out so clean. If the question said to simplify $\sqrt{11} + \sqrt{30}$, there would really be nothing you could do—that's as simple as it's going to get. The rule of thumb is that addition and square roots don't get along.

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7. $\dfrac{x+1}{x} - \dfrac{x^2-2}{x^2}$

This might look a little intimidating, but you're just subtracting fractions. How do you do that? Same as always—get a common denominator, then subtracting the new numerators. I can use x^2 as a common denominator by multiplying the term on the left by x/x:

$$\begin{aligned} \frac{x}{x} \cdot \left(\frac{x+1}{x}\right) - \frac{x^2-2}{x^2} &= \frac{x^2+x}{x^2} - \frac{x^2-2}{x^2} \\ &= \frac{(x^2+x)-(x^2-2)}{x^2} \\ &= \frac{x^2+x-x^2+2}{x^2} = \boxed{\frac{x+2}{x^2}}. \end{aligned}$$

Remember that in the first step, you have to multiply top *and* bottom by x. The reason this is allowed is because $x/x = 1$ (as long as $x \neq 0$), and multiplying by 1 doesn't actually change anything. If you forget to multiply x into the top, then you *are* changing the value, and that's no good.

8. $(m-3n)^2 \;=\; (m-3n)\cdot(m-3n) \;=\; m^2 - 3n\cdot m - m\cdot 3n + (3n)^2 \;=\; \boxed{m^2 - 6mn + 9n^2}.$

Of course, this is just the technique often called "FOIL" (First, Outer, Inner, Last). It's really the distributive property, used twice. You can save yourself a bit of time in the long run if you learn the patterns: $(a+b)^2 = a^2 + 2ab + b^2$, and $(a-b)^2 = a^2 - 2ab + b^2$. (Don't forget those 2's.)

Common mistake: Thinking $(m-3n)^2 \;=\; m^2 - (3n)^2 \;=\; m^2 - 9n^2$. (See commandment #14.) Another very common mistake is to forget parentheses and write $3n^2$ instead of $(3n)^2$, which then becomes $3^2n^2 \;=\; 9n^2$. (See commandment #13.)

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9. $\dfrac{-(-2)+(-2)^2\cdot 5}{(-2)-5} \;=\; \dfrac{2+20}{-7} \;=\; \boxed{-\dfrac{22}{7}}.$

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10. $u^3v \,+\, uv^3 \;=\; uv\cdot u^2 \,+\, uv\cdot v^2 \;=\; \boxed{uv\left(u^2+v^2\right)}.$

(Note that you **cannot** further simplify this by saying $u^2 + v^2 = (u+v)^2$; this is untrue. See commandment #14.)

A question I am often asked about this problem is, in the expression "uv^3", whether the cubing applies only to v, or to u also. It only applies to v: $uv^3 \;=\; u\cdot v^3$, not $u^3\cdot v^3$. If I wanted it to apply to u as well, I would need to indicate this with parentheses: $(uv)^3 \;=\; u^3\cdot v^3$.

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11. $w^4(w-7) \;=\; 0$

By the "Zero Product Property" (see commandment #9), this equation splits into two equations: $w^4 = 0$ and $w - 7 = 0$. The answers to the original equation are then just the solutions to either equation. (Solutions don't have to satisfy *both* $w^4 - 0$ and $w - 7 = 0$.) The second equation yields one possibility: $w = 7$. The first equation can be solved either by taking a fourth root of both sides, or by using the Zero Product Property again: $w^4 = w\cdot w\cdot w\cdot w = 0$, so $w = 0$. So the set of solutions is $\boxed{w = 0,\ 7.}$

12.
$$\begin{aligned} (x+2)(x-2) &= 13 \\ x^2 - 4 &= 13 \\ x^2 &= 17 \\ x &= \boxed{\pm\sqrt{17}}. \end{aligned}$$

Two mistakes are very often made on this problem. One is to violate commandment #9 and think that the original equation splits into the two equations $x+2 = 13$ and $x-2 = 13$. (The reasoning behind this approach is flawed. Further, you can check that the solutions to these two equations, $x = 15$ and 11, are not solutions to the original equation.) The second common mistake is to violate commandment #3 and forget the $\pm$ at the last step. If you do this, you're missing the solution $x = -\sqrt{17}$.

13. $z^3 = 5z$

There are a couple of different ways to solve this problem. Both are valid, if done correctly. If you tried it one way, make sure you understand the other approach. Each has its merits.

First approach: Bring all terms to one side and factor.

$$\begin{aligned} z^3 &= 5z \\ z^3 - 5z &= 0 \\ z(z^2-5) &= 0 \end{aligned}$$

At this point, use the Zero Product Property to split it into the two equations $z = 0$ and $z^2 - 5 = 0$. The second equation implies that $z^2 = 5$, so $z = \pm\sqrt{5}$. So we have $\boxed{z = 0 \text{ or } \pm\sqrt{5}.}$

Second approach: "Cancel" a z from each side.

$$\begin{aligned} z^3 &= 5z \\ z^2 &= 5 \end{aligned}$$

But wait! Cancellation is not pixie dust. What is the algebraic rationale for why we can cancel? We actually *divided* each side by z, and used that $z^3/z = z^2$ and $5z/z = 5$. But remember commandment #1! If I'm dividing, then need to make sure I'm not dividing by zero! The thing I'm dividing by is z itself, so **if $z = 0$, then the above cancellation is nonsense.** (This is why I don't like the word "cancellation"—it obscures what you're really doing and can make it harder to see potential problems.)

Okay, so how do we salvage this? It's not bad—we just have to treat two situations differently: there's the $z = 0$ situation and the $z \neq 0$ situation.

$z^3 = 5z$ ↗ If $z \neq 0$, then the cancellation is valid: $z^2 = 5$, so $z = \pm\sqrt{5}$.

$z^3 = 5z$ ↘ If $z = 0$, then, well, $z = 0$.

So there's sort of a fork in the road, exactly as we have with the Zero Product Property. Notice that we end with the same solutions as with the first approach.

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14. $\dfrac{3x-1}{4} = \dfrac{2x+3}{5}$

The easiest thing to do here is to multiply both sides by $4 \cdot 5 = 20$ to get rid of the denominators:

$$\begin{aligned} 20 \cdot \left(\frac{3x-1}{4}\right) &= 20 \cdot \left(\frac{2x+3}{5}\right) \\ 5 \cdot (3x-1) &= 4 \cdot (2x+3) \\ 15x - 5 &= 8x + 12 \\ 7x &= 17 \end{aligned}$$

So $\boxed{x = 17/7}$.