Some G-M-type Banach spaces and K-groups of operator algebras on them

ZHONG Huaijie, CHEN Dongxiao & CHEN Jianlan

Department of Mathematics, Fujian Normal University, Fuzhou 350007, China
Correspondence should be addressed to Zhong Huaijie (email: zhonghuaijie@sina.com.cn)
Received May 6, 2003

Abstract By providing several new varieties of G-M-type Banach spaces according to decomposable and compoundable properties, this paper discusses the operator structures of these spaces and the K-theory of the algebra of the operators on these G-M-type Banach spaces through calculation of the K-groups of the operator ideals contained in the class of Riesz operators.

Keywords: Incompoundable Banach space, radical operator Ideal, Banach algebra, K-theory.

DOI: 10.1360/02ys0372

Since the appearance of the so-called G-M-type spaces following the first heredently indecomposable Banach space $X_0$, constructed by Gowers and Maurey in ref. [1], many open questions about the structural theory of Banach spaces have been solved (see refs. [2—5]). And what is more, the new types of Banach space advance the theory of K-theory of operator algebra on Banach spaces with some spectacular achievements. It is well-known that, as $X$ is the infinite-dimensional Hilbert space $H$, $B(H)$ is a unital Banach algebra ($C^*$-algebra), and its K-group is trivial, namely $K_0(B(H)) = K_1(B(H)) = 0$. Moreover almost all the K-groups of Banach algebra of operators on the common classical Banach spaces (including function spaces $L_p, C(\Omega)$ and sequence spaces $l_p, c_0 (1 \leq p \leq \infty)$) are trivial too (see refs. [6—8], example 2.8 of ref. [9], etc.). But after people found the special operator structures on heredarily indecomposable Banach spaces $X : B(X) = \{\lambda I_X + S; \lambda \in \mathbb{C}, S \in S(X), I_X$ is the identity operator on $X, S(X)$ is strictly singular operator ideal in $B(X)\}$, Laustsen proved that $K_0(B(X)) = \mathbb{Z}^2$ by discussing $K_i(S(X)), i = 0, 1$ (see ref. [9]). Through complicate and skillful construction of a Banach space $X$ combing the G-M-type Banach space with a classical space, Zsak proved that some $K_0(B(X))$ is equal to $\mathbb{Q}$, the additive group of rationals. Together with the fact that given a pair of nonnegative integers $m$ and $n$, a Banach space $X$ can be constructed (also a mixture of hereditarily indecomposable space and some incomparable classical spaces) with $K_0(B(X)) = \mathbb{Z}^m$ and $K_1(B(X)) = \mathbb{Z}^n$ (see ref. [9]). People have a bold conjecture, that is: given two arbitrary commutative groups $G_1$ and $G_2$, there would be some Banach space $X$, with $K_0(B(X)) = G_1$ and $K_1(B(X)) = G_2$ (see ref. [8] or [9]). In a word, the new pattern of structural theory of spaces has been greatly enriching

Copyright by Science in China Press 2004
the study on the K-theory of operator algebra.

This paper includes two organic parts. The first focuses on some new kinds of G-M-type Banach spaces, the relation between all these spaces, and their operator structures. The second concerns with the K-groups of Banach algebras of all continuous linear operators on these new spaces. The main achievements can be summed up as the following theorems:

**Theorem 1.** Besides the three kinds of famous G-M-type Banach spaces (see refs. [1, 10, 11]), i.e. hereditarily indecomposable space (abbreviated as $H.I.$), quotient hereditarily indecomposable space (abbreviated as $Q.H.I.$) and quotient hereditarily indecomposable space (abbreviated as $Q.H.I_{C}.$), there are still two new kinds of G-M-type Banach spaces: quotient incompoundable space (abbreviated as $Q.I_{C}.$) and quotient indecomposable space (abbreviated as $Q.I.$). There are examples to show the independence and non-interference relations among these five G-M-type Banach spaces. Here is the relation graph:

![Relation graph of five G-M-type Banach spaces](image)

Fig. 1. Relations among the five G-M-type Banach spaces.

**Remark.** (a) Double arrow $Q.H.I \Rightarrow H.I.$ means that if $X \in Q.H.I.$, then $X \in H.I.$.; (b) single arrow $Q.H.I \rightarrow Q.H.I_{C}.$ means that there is a contrary conjugate dual relation: if $X^{*} \in Q.H.I_{C}.$, then $X \in Q.H.I.$.

**Theorem 2.** Let $X$ be a complex incompoundable space or a quotient indecomposable space. Then $B(X) = \{\lambda I_{X} + S : \lambda \in \mathbb{C}, S \in S^{c}(X)\}$, here $S^{c}(X)$ is a strictly cosingular operator ideal in Banach algebra $B(X)$.

**Remark.** Comparing with the results of ref. [1] regarding the complex hereditarily indecomposable spaces, we find that, as to the common space $X$, the strictly singular operator ideals $S(X)$ does not contain $S^{c}(X)$ in this paper, and vice versa (according to refs. [12,13]).

**Theorem 3.** For any complex infinite-dimensional Banach space $X$, the K-groups of every closed operator ideal $A(X)$ contained in the Riesz operator class $R(X)$ of $B(X)$ are definite, i.e.

$$K_{0}(A(X)) = \mathbb{Z}, \quad K_{1}(A(X)) = 0.$$

**Remark.** Ref. [9] gives only the result for $A(X) = S(X)$, but $R(X)$ includes...

www.scichina.com
many operator ideals such as \( S^c(X) \) and the radical operator ideal \( J(X) \), so Theorem 3 is a generalization of ref. [9].

**Theorem 4.** Let \( X \) be a complex infinite-dimensional Banach space. If \( B(X) = C I_X + A(X) \), in which \( A(X) \) is some closed operator ideal contained in Riesz operator class \( R(X) \), then

\[
K_0(B(X)) = \mathbb{Z}^2, \quad K_1(B(X)) = 0.
\]

Especially, using Theorem 1 and Theorem 2, we get the K-groups of Banach algebra \( B(X) \), for \( X \) = the new G-M-type Banach spaces, i.e. incompoundable space, quotient indecomposable space and so on.

1 G-M-type Banach spaces with some indecomposable properties

**Lemma** \(^{[6,11]}\). Let \( M \) and \( N \) be two closed subspaces of infinite-dimensional Banach space \( X \). Then the following assertions are equivalent: a) \( M \perp N \) in \( X \), denoted as \( M \perp N \subset X \), here the closed subspaces \( M \) and \( N \) are said to be perpendicular if \( M \cap N = 0 \) and \( M + N \) is closed in \( X \); b) the map \( i : M \otimes N \rightarrow M + N \) defined by \( i(x, y) = x + y \) is an isomorphism (isomorphisms in this paper all refer to the linear isomorphism and normed topological homeomorphism); c) \( \rho(M, N) := \inf\{\|x - y\| : x \in M, y \in N, \|x\| = \|y\| = 1\} > 0 \); d) the dual space \( X^* = M^0 + N^0 \), here the annihilator of \( M, M^0 := \{f \in X^* : \text{for each } x \in M, f(x) = 0\} \).

**Definition** \(^{[1]}\). Banach space \( X \) is indecomposable, denoted as \( X \in I_1 \), if \( X \) cannot be decomposed into topological direct sum of two infinite-dimensional and closed subspaces \( M \) and \( N \). If each infinite-dimensional closed subspace of \( X \) is indecomposable, then \( X \) is said to be hereditarily indecomposable, denoted as \( X \in H.I. \). Obviously, \( X \in H.I. \) if and only if there exist no infinite-dimensional closed subspaces \( M \) and \( N \) such that \( M \perp N \subset X \).

**Example 1.** The reflexive space \( X_G \) constructed in ref. [1] is the first H.I. space. Ref. [10] proved further that its conjugate space \( X_G^* \) is H.I. space, too. In addition, ref. [14] has constructed a reflexive H.I. space \( X \), on which there exist strict singular operators which are not compact.

**Definition 2** \(^{[10]}\). Banach space \( X \) is called quotient hereditarily indecomposable if all the QS-spaces of \( X \) are indecomposable, here the QS-space of \( X \) means \( Y/Z, Z \subset Y \subset X, \dim(Y/Z) = \infty \). The quotient hereditarily indecomposable space is denoted as \( Q.H.I. \).

Obviously by the definition, \( Q.H.I. \subset H.I. \). For the conjugate spaces, the following proposition reflects the so-called contrary conjugate dual property.

**Proposition 1** \(^{[10]}\). If \( X^* \in Q.H.I. \), then \( X \in Q.H.I. \).

**Example 2.** Ferenczi has proved Proposition 1 in ref. [10] and showed that actually G-M-type space \( X_G(X_G^*) \) is a \( Q.H.I. \) space. He has also constructed a reflexive Banach

Copyright by Science in China Press 2004
space $X_F \in H.I.$, but $X_F^* \notin H.I.$. So we can judge by Proposition 1 that $X_F$ is the first example showing that $X \in H.I. \setminus Q.H.I.$.

Now we give a new class of Banach spaces having the indecomposable property as the following.

**Definition 3.** Banach space $X$ is said to be quotient indecomposable if no infinite-dimensional quotient space of $X$ is decomposable. The class of quotient indecomposable spaces is denoted as $Q.I.$.

**Proposition 2.** $H.I.$ and $Q.I.$ properties of Banach space are contrary dual, that is:
a) if $X^* \in H.I.$, then $X \in Q.I.$; b) if $X^* \in Q.I.$, then $X \in H.I.$.

**Proof.** a) If $X \notin Q.I.$, then, by Definition 3, there exist $Y \subset X$ and quotient space decomposition $X/Y = M \oplus N$. Then we can get $X^0 \approx (X/Y)^* = M^* \oplus N^*$ in $X^*$ by Dieudonné principle, that is $M^* \perp N^* \subset X^*$, thus $X^* \notin H.I.$.

b) If $X \notin H.I.$, then, by Definition 1, there exists $M \perp N \subset X$. Write $X_0 = M \oplus N$. By Dieudonné principle again, $X^*/X_0^0 \approx X_0^* = M^* \oplus N^*$, so $X^* \notin Q.I.$.

**Example 3.** Let us come to the reflexive Banach space $X_F \in H.I.$. By Proposition 2, $X_F^* \in Q.I.$. In addition, we have shown $X_F^* \notin H.I.$ in Example 2, thus $X_F^* \notin Q.H.I.$.. All these make it clear that $Q.I.$ is really a new space class different from $Q.H.I.$ and $H.I.$.

2 G-M type Banach spaces with some indistinguishable properties

If Banach space $X$ contains two infinite-codimensional subspaces $M$ and $N$ (dim$X/M = \dim(X/N) = \infty$) and $M + N = X$, we say that $X$ is compounded by $M$ and $N$. It is denoted as $M \vee N \supset X$.

In as early as the 1970s, people asked a basic question of whether every Banach space is compoundable.

**Lemma 2[6,11].** As to two infinite-codimensional subspaces $M$ and $N$ of Banach space $X$, the following statements are equivalent: a) $M \vee N \supset X$; b) the map $Q : X \to X/M \oplus X/N$ defined by $Qx = (Q_Mx, Q_Nx)$ is a surjective map, here $Q_M$ refers to quotient map $X \to X/M$, $Q_N$ refers to $X \to X/N$; c) in $X^*$, $M^* \perp N^*$.

**Definition 4.** Banach space $X$ is said to be quotient indistinguishable, if every (infinite-dimensional) quotient space of $X$ is indistinguishable. The class of quotient indistinguishable spaces is denoted as $Q.I_C$.

**Definition 4'.** Banach space $X$ is said to be indistinguishable if there exists no $M \vee N \supset X$. The class of indistinguishable spaces is denoted as $I_C$.

It is obvious that $Q.I_C. \subset I_C$. by the definition. On the other hand, in fact it is easy to show $Q.I_C. = I_C$. In this paper we prefer to use the sign $Q.I_C.$ in Definition 4 because
it is more convenient to compare the space with $H.I.$.

**Proposition 3.** $H.I.$ and $Q.I_C.$ properties of Banach spaces are contrary dual, that is: a) if $X^* \in H.I.$, then $X \in Q.I_C.$; b) if $X^* \in Q.I_C.$, then $X \in H.I.$.

**Proof.** a) Suppose $X \not\in Q.I_C.$, then $M \vee N \supset X/Y$. According to Lemma 2, $M^0 \perp N^0$ in $Y^0 \simeq (X/Y)^*$. We find that $Y^0, M^0$ and $N^0$ are all (infinite-dimensional) subspaces of $X^*$. It follows that $X^* \not\in H.I.$.

b) If $X \not\in H.I.$, then there exists $M \perp N \subset X$. According to Lemma 1, $M^0 \vee N^0 \supset X^*$, so $X^* \not\in Q.I_C.$.

**Example 4.** As we know, G-M spaces $X_G$ and $X_G^*$ are both reflexive $Q.H.I.$ (also $H.I.$) ones, so according to Proposition 3 they are the first examples of $Q.I_C.$ spaces, too. Here we would emphasize that $Q.I_C.$ and $H.I.$ are two independent classes of spaces and they cannot be contained each other. Let us take the reflexive spaces $X_F \in H.I.$ and $X_F^* \not\in H.I.$ as mentioned earlier; according to Proposition 3, we will soon find $X_F \in H.I. \setminus Q.I_C.$ and $X_F^* \in Q.I_C. \setminus H.I.$.

**Definition 5.** Banach space $X$ is a quotient hereditarily incompoundable space if all its infinite dimensional quotient spaces are hereditarily incompoundable, or equivalently, if all its QS-spaces are incompoundable according to Definition 3. The class of quotient hereditarily incompoundable spaces is denoted as $Q.H.I.C.$.

**Definition 5'.** Banach space $X$ is hereditarily incompoundable if all its subspaces are incompoundable. The class of hereditarily incompoundable spaces is denoted as $H.I.C.$.

It is easy to see that $Q.H.I.C. = H.I.C.$. In this paper, we prefer to use Definition 5 because it is more convenient to compare it with $Q.H.I.$ in Definition 2 in form. But sometimes we also use $H.I.C.$ of Definition 5' because it is more convenient to compare it with $Q.I.$ in Definition 3 in form (see fig. 1).

**Proposition 4.** $Q.H.I.$ and $Q.H.I.C.(=H.I.C.)$ properties of Banach spaces are contrary dual, that is: a) if $X^* \in Q.H.I.$, then $X \in Q.H.I.C.$; b) if $X^* \in Q.H.I.C.$, then $X \in Q.H.I.$.

**Proof.** Similar to the proofs of the propositions above. Omitted here.

**Corollary 1.** If $X^* \in Q.H.I.C.(=H.I.C.)$, then $X \in Q.I.$.

**Remark.** Example 3 has shown that reflexive Banach space $X_F^* \in Q.I.$, and $X_F^* \not\in Q.H.I.$, so that by Proposition 4, $X_F = X_F^* \not\in Q.H.I.C.$.. This means that the reverse of Corollary 1 is not true. Similarly the positions of $Q.H.I.C.$ and $Q.I.$ in the corollary cannot be exchanged.

It is interesting that there is the following result similar to Proposition 1:

**Proposition 5.** $Q.H.I.C.$ property of Banach space itself is contrary dual. Namely,
if $X^* \in Q.H.I_C$, then $X \in Q.H.I_C$.

**Proof.** Supposing $X \not\in Q.H.I_C$, there is a QS-space $Z/Y$ of $X$, with subspace $Y \subset Z \subset X$ and $\dim(Z/Y) = \infty$; $Z/Y$ is compoundable, that is $(M/Y) \cap (N/Y) \supset Z/Y$. Now according to Lemma 2, $(M/Y)^0 \perp (N/Y)^0 \subset Y^0/Z^0$ and $Z^0 \subset Y^0 \subset X^*$ in the dual space $(Z/Y)^* \approx Y^0/Z^0$. So let $W := (M/Y)^0 + (N/Y)^0 \subset Y^0/Z^0$. As a subspace of QS-space $Y^0/Z^0$ of $X^*$, $W$ is a QS-space of $X^*$. It is therefore compoundable, i.e. $(M/Y)^0 \cap (N/Y)^0 \supset W$. It contradicts the condition $X^* \in Q.H.I_C$.

**Corollary 2.** If $X^* \in Q.H.I_C$, then $X \in Q.I$.

**Remark.** The reverse of Corollary 2 is not true. Because $X_F$ is a reflexive $H.I.$ space, according to Proposition 3, $X_F^*$ is a $Q.I.C.$ space. But we have shown that $X_F = X_F^*$ is not a $Q.H.I_C$ space in the previous remark.

Synthesizing Propositions 1, 4 and 5, we have the following

**Corollary 3.** For the reflexive Banach space $X$, the following statements are equivalent: a) $X \in Q.H.I.$; b) $X^* \in Q.H.I.$; c) $X \in Q.H.I.C.$; d) $X^* \in Q.H.I.C.$.

**Example 5.** From the reflexive $Q.H.I.$ space $X_0(X_0^*)$, we can get an example of $Q.H.I.C.$ space $X_0^*(X_0)$ immediately from Corollary 3.

**Example 6.** The definition shows that $Q.H.I.C.$ space must be $Q.I.C.$ space. There are examples showing that the reverse statement is not true: $X_F^*$ is a $Q.I.C.$ space (according to Example 4), but $X_F^*$ is not a $Q.H.I.C.$ space (according to Example 2 and Corollary 3).

**Example 7.** It is obvious that $X_F \in H.I.\setminus Q.H.I_C$.

**Example 8.** It is obvious that $X_F^* \in Q.I.\setminus Q.H.I_C$.

**Remark.** Examples above show not only the existence of $Q.H.I.C.$ spaces, but also the difference between $Q.H.I.C.$ and other G-M-type spaces. But we still cannot tell whether there are spaces $X \in Q.H.I.C.\setminus H.I.$, or space $X \in Q.H.I.C.\setminus Q.I.$

In addition, although we have known that $Q.H.I.$ space itself has the contrary dual property in Proposition 1 and $Q.H.I.C.$ space itself has the contrary dual property in Proposition 5 too, we notice that the $Q.I.$ and $Q.I.C.$ spaces, as well as the $H.I.$ space (known before), themselves do not have contrary dual property, e.g. $X_F^* \in Q.I.$, but $X_F \not\in Q.I.$.

By now, we have finished the proof of the relations within the five G-M-type Banch spaces in Theorem 1 and fig. 1.

### 3 Operator structures of some G-M-type spaces

Let $X$ be an infinite-dimensional complex Banach space and $B(X)$ be a complex Banach algebra of continuous linear operators on $X$. There is a special operator class
$R(X)$—Riesz operator class in $B(X)$. For common Banach spaces, the sum and multiplication of two Riesz operators $T_1, T_2 \in R(X)$ may not be in $R(X)$ (see the examples in ref. [15], §3). So usually $R(X)$ is not an operator ideal in $B(X)$. But $R(X)$ contains ideals such as compact operator ideal, strictly singular operator ideal, strictly cosingular operator ideal and so on. For the necessity of the following discussions, we display the characteristics of $R(X)$ as follows (see refs. [15—17]):

**Proposition 6.** Let $X$ be an infinite-dimensional complex Banach space. Then the following assertions are equivalent: a) $T \in R(X)$, that is, $\pi(T)$ is a quasi-nilpotent element of Calkin algebra $C(X)$, where $\pi : B(X) \rightarrow C(X) := B(X)/K(X)$ is the canonical homomorphism; b) $T$ has the same spectral structure in $B(X)$ as a compact operator. Namely, $\sigma(T)$, spectrum of $T$, is a countable set, $\sigma_0(T) = \{0\}$, for any $0 \neq \lambda \in \sigma(T)$, $\lambda$ is an isolated point of $\sigma(T)$, and the spectral projective space, corresponding to $\lambda := \text{Im}(E(\lambda; T))$, the range of spectral projective operator $E(\lambda; T) := \{y \in X : \text{there exists } x \in E, E(\lambda; T)x = y\}$ is finite-dimensional; c) for any $\lambda \neq 0$, $\lambda I_X - T$ is a Fredholm operator, that is, $\lambda I_X - T \in \Phi(X) := \{A \in B(X) : \text{Im}A \text{ is closed, } \text{dim Ker}A < \infty \text{ and codim Im}A < \infty\}$; d) for any $\lambda \neq 0$, $\lambda I_X - T$ is a Fredholm operator with zero index, that is, $\lambda I_X - T \in \Phi_0(X) := \{A \in B(X) : A \in \Phi(X) \text{ and index of } A, \text{ind}(A) := \text{dim Ker}(A) - \text{codim Im}A = 0\}$.

Besides compact operator ideal, there are common operator ideals such as $S(X)$ and $S^c(X)$ also contained in the class of Riesz operators $R(X)$.

The strictly singular operator ideal $S(X) := \{T \in B(X) : \text{the restriction of } T : Z \rightarrow X \text{ is not an isomorphism, here } Z \text{ is an arbitrary infinite-dimensional closed subspace of } X\}$.

The strictly cosingular operator ideal $S^c(X) := \{T \in B(X) : \text{for every infinite-dimensional quotient mapping } Q : X \rightarrow Y, \text{the composite operator } QT : X \rightarrow Y \text{ is not surjective }\}$.

**Lemma 3**. Let $X$ and $Y$ be two infinite-dimensional Banach spaces. For operator $T \in B(X, Y)$, if $T \not\in \Phi(X, Y)$ and $T \not\in S(X, Y)$, then there exist two infinite-dimensional closed subspaces $M, N$ of $X$ perpendicular to each other, $M \perp N \subset X$, where the left semi-Fredholm operator class $\Phi_+(X, Y) := \{T \in B(X, Y) : \text{Im}T \text{ is closed in } X \text{ and dim } \text{Ker}(T) < \infty\}$.

**Proposition 7.** Let $X$ be a complex $H.I.$ space. Then (i) $B(X) = \{\lambda I_X + S : \lambda \in \mathbb{C}, S \in S(X)\}$; (ii) $R(X) = S(X)$, and the class of Riesz operators is the biggest nontrivial operator ideal in Banach algebra $B(X)$.

**Proof.** In ref. [11], there is a much simpler proof of conclusion (i) than ref. [1] by using Lemma 3.

Now we prove $R(X) = S(X)$. Since $R(X) \supset S(X)$ (ref. [12] 26.7.3), we need to prove $R(X) \subset S(X)$. That is easy, when $T \in R(X)$, we have $0 \in \sigma_0(T), T \not\in \sigma(T)$.
\( \Phi_+(X) \). Notice that \( H.I. \) space \( X \) does not contain infinite-dimensional closed subspaces perpendicular to each other, so \( T \in S(X) \).

The conception of Kato spectrum (ref. [18]) of an operator \( T \) is to be used here and in section 4. That is, \( \sigma_+(T) := \{ \lambda \in \mathbb{C}: \text{either Im}(\lambda I_X - T) \text{ is nonclosed, or Im}(\lambda I_X - T) \text{ is closed but dim Ker}(\lambda I_X - T) = \text{codim Im}(\lambda I_X - T) = \infty \} \). The definition implies that \( \sigma_+(T) \subset \sigma_+(T) := \{ \lambda \in \mathbb{C}: (\lambda I_X - T) \notin \Phi(X) \} \) and \( \sigma_+(T) \neq \emptyset \), so for \( T \in R(X) \), \( 0 \in \sigma_+(T) \).

Since \( R(X) = S(X) \), we have verified that \( R(X) \) becomes an operator ideal in \( B(X) \). To prove that it is the biggest one, we need to demonstrate that in \( B(X) \) if operator ideal \( A(X) \neq B(X) \), then there should be \( A(X) \subset R(X) = S(X) \). Suppose \( T \in A(X) \setminus R(X) \), then by conclusion (i) of this proposition, we have the expression \( T = \lambda I_X + S \) and \( \lambda \neq 0 \). It follows from (d) in proposition 6 that \( \lambda I_X + S \in \Phi_0(X) \). So there is a finite-rank operator \( F \in F(X) \) such that \( T + F \) is a convertible operator. But every operator ideal should contain \( F(X) \), so \( T + F \in A(X) \). Now the identity operator \( I_X = (T + F)(T + F)^{-1} \in A(X) \), this comes to a contraction \( A(X) = B(X) \). The supposition \( T \in A(X) \setminus R(X) \) is not true, so \( A(X) \subset R(X) \) is proved.

**Corollary 4.** Since \( Q.H.I. \subset H.I. \), when \( X \) is a complex \( Q.H.I. \) space, the conclusion corresponding to Proposition 7 holds.

**Lemma 4** \( ^6 \). Let \( X \) and \( Y \) be two infinite-dimensional Banach spaces, \( T \in B(X, Y) \), \( T \notin \Phi_-(X, Y) \) and \( T \notin S^\ast(X, Y) \). Then space \( Y \) is compendiable, where the right semi-Fredholm operator class \( \Phi_-(X, Y) := \{ T \in B(X, Y) : \text{Im}(T) \text{ is closed and codim Im}(T) < \infty \} \).

**Proof.** Because we notice that the proof in ref. [5] has an oversight, so we give a new strict proof. The oversight is that, for a closed (by norm) subspace \( N \) of conjugate space \( Y^* \), its biannihilator \( (0 \ N^0) = \overline{N}^\ast \), the \( w^\ast \)-topological closure of \( N \), but usually \( (0 \ N^0) \neq N \) (unless \( Y \) is a reflexive space).

We first show the fact that, if \( T \in B(X, Y) \) and \( T^\ast \notin \Phi_+(Y^*, X^*) \), then for any \( \varepsilon > 0 \), there exists a weak \( \ast \)-closed (infinite-dimensional) subspace \( N^\ast \subset Y^* \) such that the restriction of operator \( T^\ast \) to \( N^\ast \), \( T^\ast |_{N^\ast} \), is a compact operator, and \( \| T^\ast |_{N^\ast} \| \leq \varepsilon \).

We can construct by induction two bi-orthogonal sequences \( \{ y_n \} \subset Y \) and \( \{ y_n^\ast \} \subset Y^* \) such that, for any \( i, j = 1, 2, \cdots \),

\[
y_j^\ast(y_i) = \delta_{ij}, \quad \| y_j^\ast \| = 1, \quad \| y_i \| \leq 2^{k-1}, \quad \| T^\ast y_j^\ast \| < 2^{-3j}\varepsilon.
\]

The existence of \( y_1 \) and \( y_1^\ast \) is due to \( T^\ast \notin \Phi_+(Y^*, X^*) \). So the action of \( T^\ast \) on the unit sphere \( S(Y^*) \) is not bounded below. Assume that there have been \( y_1, \cdots, y_k \) and \( y_1^\ast, \cdots, y_k^\ast \) conforming to the request. Let \( Z_k = (\text{span}\{ y_1, \cdots, y_k \})^0 \subset Y^* \). Since \( T^\ast \notin \Phi_+(Y^*, X^*) \) and the codimension of \( Z_k \) is finite, \( T^\ast \) on the unit sphere \( S(Z_k) \) is still boundless below, so there is a \( y_{k+1}^\ast \in Z_k \) such that \( \| y_{k+1}^\ast \| = 1 \) and \( \| T^\ast y_{k+1}^\ast \| \leq 2^{-3(k+1)}\varepsilon \). Take a \( y_{k+1} \in Y \) again such that \( \| y_{k+1} \| < 2 \) and \( y_{k+1}^\ast(y_{k+1}) = 1 \). Let

www.scichina.com
\[ y_{k+1} = g_{k+1} - \sum_{i=1}^{k} y_i^*(g_{k+1})y_i, \] then we have \( y_i^*(y_j) = \delta_{ij}, i, j = 1, 2, \ldots, k + 1. \) And we also have
\[
\| y_{k+1} \| \leq \| g_{k+1} \| (1 + \sum_{i=1}^{k} \| y_i^* \| \| y_i \|) \leq 2(1 + \sum_{i=1}^{k} 2^{2i-1}) \leq 2^{2k+1} = 2^{2(k+1)-1}.
\]
Now define an operator \( A \in B(Y^*, X^*) \) as follows:
\[
Ay^* = \sum_{k=1}^{\infty} y^*(y_k)T^*y_k^* \quad (y^* \in Y^*).
\]
Inequalities \( \| T^*y_k^* \| \leq 2^{-3k-2} \varepsilon \) and \( \| y_k \| \leq 2^{2k-1} \) imply that \( A \) is the limit of a finite-rank operator sequence, so \( A \) is a compact operator and \( \| A \| < \varepsilon. \)

Furthermore, let \( B \in B(X, Y) \) be as follows:
\[
Bx = \sum_{k=1}^{\infty} T^*y_k^*(x)y_k, \ x \in X.
\]
Then it is obvious that \( B^* = A. \)

Therefore, \( T^* - A = (T - B)^* \) is a conjugate operator and it is continuous under the normed topology or the weak \* topology on \( Y^*. \) Thus \( N' = \text{Ker}(T^* - A) \) is a weak \* closed subspace in \( Y^*. \) Obviously, for any \( k, y_k \in N'. \) So \( N' \) is infinite-dimensional. Since \( T^* = A \) on \( N' \), \( T^* \) is compact on \( N' \) and \( \| T^* \|_{N'} = \| A \| < \varepsilon. \)

Now we turn to the new proof of Lemma 4. Since \( T \not\in S^0(X, Y) \), there is an infinite-dimensional quotient map \( Q_M : Y \longrightarrow Y/M \) such that \( Q_MT : X \longrightarrow Y/M \) is surjective. Since \( (Q_MT)^* = T^*Q_M^* : (Y/M)^* = M^0 \longrightarrow X^* \) is an injective isomorphism, there is \( c > 0 \) such that
\[
\| T^*y^* \| \geq c \| y^* \|, \text{ for every } y^* \in M^0. \quad (3.1)
\]
For \( T \not\in \Phi^+(X, Y) \), equivalently, \( T^* \not\in \Phi^+(Y^*, X^*). \) By the fact proved above, there exists an infinite-dimensional weak \* closed subspace \( N' \subset Y^* \) such that
\[
\| T^*y^* \| \leq \frac{c}{2} \| y^* \|, \text{ for every } y^* \in N'. \quad (3.2)
\]
Up to now, analogous to Lemma 3, we have obtained two subspaces of \( Y^*: M^0 \perp N' \subset Y^* \) by formulae (3.1) and (3.2). Define the annihilator of \( N' \) as \( N = [0(N')] \), then \( N^0 = [0(N')]^0 = \text{the weak \* closure of } N' = N'. \)

Finally, from \( M^0 \perp N', \ i.e. \ M^0 \perp N^0 \subset Y^* \), we get \( M \lor N = Y \) by Lemma 2. (The verification that \( M \) and \( N \) are infinite-codimensional is omitted.)

**Proposition 8.** Let \( X \) be a complex \( Q.H.I.C.(= I.C.) \) space. Then (i): \( B(X) = \{ M_X + S : \lambda \in \mathbb{C}, S \in S^0(X) \}; \) (ii): \( R(X) = S^0(X) \), and the class of Riesz operators is the maximal nontrivial operator ideal.

**Proof.** The proof is omitted because it is almost parallel to the proof of Proposition 7, except that Lemma 4 should be used in the proof of (i).

**Corollary 5.** For the complex \( Q.H.I.C.(= H.I.C.) \) space \( X, B(X) = \{ \lambda I_X + S : \)
G-M-type spaces & K-groups of operator algebras

\( \lambda \in \mathbb{C}, S \in S^c(X) \) and \( R(X) = S^c(X) \) is the maximal nontrivial operator ideal in \( B(X) \).

**Proof.** The proof is omitted because it is almost parallel to the proof of Proposition 7, except that Lemma 4 should be used in the proof of (i).

**Proposition 9.** Let \( X \) be a complex Q.I. space. Then (i): \( B(X) = \{ \lambda I_X + S : \lambda \in \mathbb{C}, S \in S^c(X) \} \); (ii): \( R(X) = S^c(X) \), and the class of Riesz operators is the maximal nontrivial operator ideal.

**Proof.** For any \( T \in B(X) \), the right essential spectrum of \( T, \sigma^e_r(T) := \{ \lambda \in \mathbb{C} : \lambda I_X - T \notin \Phi_+(X) \} \) is nonempty. So we can take a \( \lambda \in \mathbb{C} \) such that \( \lambda I_X - T \notin \Phi_+(X) \). Now we verify the fact that \( \lambda I_X - T \in S^c(X) \), which implies (i). Otherwise, by Lemma 4, there exist two infinite-codimensional subspaces \( M \) and \( N \) such that \( M + N = X \). Let \( X_0 = M \cap N \), thus \( X_0 \) is infinite-codimensional in \( X \). Now the infinite-dimensional quotient space \( X/X_0 = (M/X_0) \oplus (N/X_0) \) is a decomposition of direct sum, that is, \( (M/X_0) \perp (N/X_0) \subset X/X_0 \). This contradicts the fact that \( X \) is a quotient indecomposable space.

The proof of Proposition 9(ii) is analogous to the two preceding propositions, and hence omitted.

Using the dual method analogous to the proof of Proposition 7(i), we get the following result about the operator structure on two classes of G-M-type Banach spaces better than the Corollary 4 of Proposition 7 and Corollary 5 of Proposition 8:

**Corollary 6.** When \( X \) is a Q.H.I. space or Q.H.J.C. (= H.J.C.) space, \( B(X) = CI_X + R(X) \), and \( R(X) = S(X) = S^c(X) \) is the maximal nontrivial operator ideal in \( B(X) \). Up to now, we have finished the proof of Theorem 2.

4 K-groups of some operator ideals

For a given Banach space \( X \), the Banach algebra of all continuous linear operators is denoted by \( B(X) \). It is also a normed ring. The conception of an ideal formed by a subring of \( B(X) \) is familiar. When \( X \neq Y \), since we need to discuss in the following sections some collections of subspaces of Banach space \( B(X, Y) \), denoted by \( A(X, Y) \), we require the conception of generalized ideals.

**Definition**[12]. Let \( \mathcal{A} \) be the class of all operators between arbitrary Banach spaces, and \( A \) is a subclass of \( \mathcal{A} \). We say that \( A \) is an operator ideal if \( A \) satisfies the following conditions:

(i) \( A \) contains all finite-rank operators. That is, for any two Banach spaces \( X \) and \( Y \), we have \( A \supset F(X, Y) \).

(ii) For any Banach space \( X \), the branch \( A \cap B(X) := A(X) \) of \( A \) is an ideal of operators in \( B(X) \).
(iii) The element of $A$ has the property of multiplicative absorbency: if $T \in A \cap B(X, Y) = A(X, Y)$, $Q \in B(Y, Z)$ and $R \in B(W, X)$, then $QTR \in A(W, Z)$.

**Definition 7**[12, 19]. As a branch of radical operator ideal $J$, $J(X, Y) := \{ T \in B(X, Y) : \forall A \in B(X, Y), I_X - AT \in \Phi(X) \} = \{ T \in B(X, Y) : \forall B \in B(Y, X), I_Y - TB \in \Phi(Y) \}$.

Here, the identity operators on $X$ and $Y$ are denoted by $I_X$ and $I_Y$, respectively.

For a given Banach space $X$, the branch in $B(X)$ of the radical operator ideal $J(X)$, which is a closed ideal of $B(X)$, has many characteristics (and therefore has many other names). Because it is important to the following discussions, we introduce the following proposition (the detailed proof is in refs. [12, 13]).

**Proposition 10.** a) $J(X) = \{ T \in B(X) : \text{under the canonical homomorphism } \pi : B(X) \to C(X) := B(X)/K(X), \pi(T) \text{ is an element of Jacobson radical in Calkin algebras} \}$ (this is why we call $J$ a radical operator ideal).

b) $T \in J(X) \iff \forall A \in B(X), AT \in R(X)$
   $\iff \forall A \in B(X), TA \in R(X)$
   $\iff \forall A \in B(X), I \pm AT \in \Phi(X)$
   $\iff \forall A \in B(X), I \pm TA \in \Phi(X)$
   $\iff \forall A \in R(X), T + A \in R(X)$
   $\iff \forall A \in \Phi(X), T + A \in \Phi(X)$.

The last characteristic is why $J(X)$ is also called “the perturbation class associated with the set of Fredholm operators” (see ref. [13]).

c) For $T \in J(X)$ and every $A \in B(X)$, $A$ and $A + T$ have the same essential spectrum, that is, for $\lambda \in \mathbb{C}$, $\lambda I_X - A \in \Phi(X)$ if and only if $\lambda I_X - (A + T) \in \Phi(X)$.

d) For $T \in J(X)$, suppose $A \in \Phi_0(X)$, then $A + T \in \Phi_0(X)$.

e) $J(X)$ is the maximal ideal of operators contained in the class of Riesz operators, that is, for any operator ideal $A(X) \subseteq R(X)$, we have $A(X) \subseteq J(X)$. We also have the following inclusive relations (it usually is not the equal relation; there is no inclusive relation between $S$ and $S^c$):

\[
\supseteq S(X) \\
B(X) \supseteq R(X) \supseteq J(X) \quad \supseteq K(X) \supseteq F(X) \supseteq F(X). \\
\supseteq S^c(X)
\]

Now, we have recognized many operator ideals contained in the class of Riesz operators and it is evident that these ideals, except $J$, can be extended to generalized ideals according to their definitions in $B(X)$. 

Copyright by Science in China Press 2004
Our following work is to improve the work in ref. [9]. Instead of computing the K-groups of the ideals contained in $S(X)$, we have calculated the K-groups of the ideals contained in $R(X)$. The work in ref. [9] is based on Proposition 2.c.13. in ref. [20], namely the diagonalization of projections, by Edelstein and Wojtaszczyk. Because they only considered the case of $S(X)$, so we must rebuild the results in details.

Here, we require a fact of the spectral theory of linear operators.

**Lemma 5.** Let $X$ be an infinite-dimensional complex Banach space, $T \in B(X)$. If $\sigma_e(T)$, the essential spectrum of $T$, is a finite set, then $\sigma(T)$, the spectrum of $T$, is a countable set, and the set $\sigma(T) \setminus \sigma_e(T)$ contains isolated points in $\sigma(T)$.

**Proof.** This is shown in ref. [18], IV, Theorem 5.33.

**Proposition 11.** Let $J$ be the radical operator ideal, $X = Y \oplus Z$, $Y$ and $Z$ are two complex Banach spaces, the projection $P \in B(X)$ has a matrix representation $P = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $B \in J(Y, X), C \in J(X, Y)$. Then there exists an automorphism $U$ of $X$, a projection $P_1 \in B(Y)$, and a projection $P_2 \in B(Z)$, such that the range $\text{Im}(UP) = \text{Im}(P_1 \oplus P_2)$, or $U M = Y_0 \oplus Z_0$, where $M = \text{Im}(P)$, $Y_0 = \text{Im}(P_1)$, $Z_0 = \text{Im}(P_2)$.

**Proof.** 1) Suppose $\dim M = \text{codim} M = \infty$ in $X$ (otherwise, it is easy to obtain the result), then the spectrum and the essential spectrum of $P$ are equal and composed of 0 and 1, namely, $\sigma(P) = \sigma_e(P) = \{0, 1\}$.

2) Let $S := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \in B(X)$. Since $B, C \in J, J$ is an ideal of operators, hence $S \in J(X)$. Define $Q := P - S = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, then $\sigma_e(Q) = \{0, 1\}$ by Proposition 10(c) and it is easy to verify that $\sigma(Q)$ is a countable set by Lemma 5, every point in $\sigma(Q) \setminus \{0, 1\}$ is an isolated eigenvalue of $Q$ with finite algebraic multiplicity.

We select a closed simple curve $\Gamma$ contained in $\rho(Q)$ such that number 1 is in the interior and 0 is in the exterior of $\Gamma$ to form the spectral projection $P_0 := -(2\pi i)^{-1} \int_{\Gamma} (z - Q)^{-1} dz$.

Then it is easy to show that $P_0 Q = Q P_0, P_0 = \begin{pmatrix} A_1 & 0 \\ 0 & D_1 \end{pmatrix}$.

We also notice that $P = -(2\pi i)^{-1} \int_{\Gamma} (z - P)^{-1} dz$.

For every $z \in \Gamma, (z - P) - (z - Q) = Q - P = -S \in J$, so we can easily have via computation that

$$(z - P)^{-1} - (z - Q)^{-1} = (z - Q)^{-1}((z - Q) - (z - P))(z - P)^{-1} \in J.$$
Thus \( P_0 - P = -(2\pi \imath)^{-1} \int_{J} ((z - Q)^{-1} - (z - P)^{-1}) \, dz \in J \) (\( J \) is a closed ideal).

Now, we define \( V := I_X - P + P_0 \), then \( V \in \Phi_0(X) \) and \( V(\text{Im}(P)) = P_0 M \subset \text{Im}(P_0) \).

3) We shall show next that \( X_1 := \text{Im}(P_0) \cap \text{Ker}(P) \) is a finite-dimensional subspace of \( X \). Obviously, \( X_1 \) is an invariant subspace of \( P_0 - P \). In fact, the restriction of \( P_0 - P \) to \( X_1 \) is an identity operator, \((P_0 - P)|_{X_1} = I_{X_1}\). Notice that \((P_0 - P) \in J(X) \subset R(X)\), and the restriction of Riesz operator \( P_0 - P \) to its invariant subspace is a Riesz operator again (ref. [15], §3). So \((P_0 - P)|_{X_1} = I_{X_1} \in R(X_1)\), and hence \( \dim X_1 < \infty \).

4) We shall show now that \( VM = V \text{Im}(P) \) is a subspace of finite-codimension in \( \text{Im}(P_0) \), \( \dim (\text{Im}(P_0)/VM) < \infty \). Otherwise, since \( V \in \Phi_0(X) \), \( \text{Im}(V) \) is a subspace of finite-codimension in \( X \) and hence in \( \text{Im}(P_0) \). Now \( V \text{Im}(P) \) is an infinite-codimensional subspace in \( \text{Im}(P_0) \), and \( X = \text{Ker}(P) \oplus \text{Im}(P) \), hence there exists an infinite-dimensional closed subspace \( W \subset \text{Ker}(P) \) such that \( VW \subset \text{Im}(P_0) \).

Notice that \( W \subset \text{Ker}(P) \), so for every \( x \in W \), \( Vx = (I_X - P + P_0)x = x + P_0x \), that is, \( x = Vx - P_0x \in \text{Im}(P_0) \). Hence \( W \subset \text{Ker}(P) \cap \text{Im}(P_0) = X_1 \), but this contradicts \( \dim X_1 < \infty \) in 3).

5) Let \( N_0 = \text{Ker}(V) \cap M \), then \( N_0 \subset M \) and \( \dim N_0 \leq \dim \text{Ker}(V) < \infty \). We put \( M_1 = M \oplus N_0 \). Then \( M_1 \) is a finite-dimensional closed subspace in \( M \), and \( V \) isomorphically maps \( M_1 \) into \( \text{Im}(P_0) = A_1 Y \oplus D_1 Z \). Write \( VM_1 = Y_1 \oplus Z_1 \subset A_1 Y \oplus D_1 Z \).

Notice that \( \dim(Y/Y_1) = \infty \) or \( \dim(Z/Z_1) = \infty \). In fact, \( P \) and \( P_0 \) have the same essential spectrum, \( X/\text{Im}(P_0) \) is infinite-dimensional, so is \( Y/A_1 Y \) or \( Z/D_1 Z \) and hence \( Y/Y_1 \) or \( Z/Z_1 \) is infinite-dimensional. We suppose that \( Y/Y_1 \) is infinite-dimensional.

\[
X = Y \oplus Z = \text{Im}(P) \oplus \text{Ker}(P) = M \oplus \text{Ker}(P) = (N_0 \oplus M_1) \oplus \text{Ker}(P) = N_0 \oplus (M_1 \oplus \text{Ker}(P)) \text{.}
\]

There is a finite-rank operator \( L \) on \( X \) such that \( \text{Ker}(L) = M_1 \oplus \text{Ker}(P) \), \( L \) isomorphically maps \( N_0 \) to a finite-dimensional subspace of \( Y \) which does not intersect \( Y_1 \). And \( V + L \) is also a Fredholm operator of index 0 such that \( \text{Ker}(V + L) \cap M = \{0\} \).

6) Finally, we construct a finite-rank operator \( K \) such that \( M \subset \text{Ker}(K) \) and \( U := V + L + K \) is a bijection. Let \( Y_0 = Y_1 \oplus \text{Im}(L) \) and \( Z_0 := Z_1 \), it is easy to show that \( U \) is an isomorphism from \( M \) onto \( Y_0 \oplus Z_0 \), that is, \( UM = Y_0 \oplus Z_0 \).

Recall that \( Y_0 \) and \( Z_0 \) are the complemented subspaces of \( Y_1 \) and \( Z_1 \), respectively. \( VM = V \text{Im}(P) \) is a finite-codimensional subspace in \( \text{Im}(P_0) = A_1 Y \oplus D_1 Z \) and \( VM = Y_1 \oplus Z_1 \), \( Y_1 \subset A_1 Y \), \( Z_1 \subset D_1 Z \), therefore \( Y_1 \) and \( Z_1 \) are complemented in \( A_1 Y \) and \( D_1 Z \) respectively. Let \( P_1 \) be the projection from \( A_1 Y \) to \( Y_1 \), \( P_2 \) be the projection from \( D_1 Z \) to \( Z_1 \), \( P_1' = P_1 A_1 \) and \( P_2 = P_2 D_1 \). Then \( Y_1 = P_1' Y \) and \( Z_0 := Z_1 = P_2 Z \) are complemented in \( Y \) and \( Z \) respectively. Lastly, since \( Y_0 = Y_1 \oplus \text{Im}(L) \), \( Y_1 \) is a complemented subspace in \( Y \) and \( \text{Im}(L) \) is a finite-dimensional space, so \( Y_0 \) is a complemented subspace. Hence, there exists a \( P_1 \in B(Y) \), a projection from \( Y \) to \( Y_0 \),
$P_1 Y = Y_0$. This completes the proof of Proposition 11.

**Corollary 7.** Let $X = Y_1 \oplus Y_2 \oplus \cdots \oplus Y_n, (n \geq 2)$, $\{Y_i\}_{i=1}^n$ be complex Banach spaces. We identify the projection $P \in B(X)$ with its matrix representation $P = (a_{ij})_{i,j=1}^n$. If $A_{ij} \in A(Y_i, Y_j)$, $i \neq j$, where $A$ is a generalized ideal and $A(X) \subset B(X)$, then there exists an automorphism $U$ of $X$ and projections $P_i \in B(Y_i)$ $(i = 1, \cdots, n)$ such that $U P(X) = P_1 Y_1 \oplus P_2 Y_2 \oplus \cdots \oplus P_n Y_n$.

Corollary 7 is a generalization of Proposition 11. We can adopt almost the proof of diagonalization theory by Edelstein and Wojtaszczyk. Only minor adjustments are necessary to ensure that the number of spaces is $n$. So the proof is omitted.

**Corollary 8.** Let $X = Y_1 \oplus Y_2 \oplus \cdots \oplus Y_n$ $(n \geq 2)$ and any two distinct spaces in the family $\{Y_i\}_{i=1}^n$ be essentially incomparable. Then for any projection $P \in B(X)$, there exists an automorphism $U$ of $X$ and projections $P_i \in B(Y_i) (i = 1, \cdots, n)$ such that $U P(X) = P_1 Y_1 \oplus P_2 Y_2 \oplus \cdots \oplus P_n Y_n$.

Corollary 8 is a typical application of Corollary 7. In fact, the definition of essential incomparable spaces is just that $B(Y_i, Y_j) = J(Y_i, Y_j)$ for $i \neq j$ (see ref. [19], etc.), so we have $A_{ij} \in J(Y_i, Y_j), i, j = 1, 2, \cdots, n, i \neq j$.

For discussing $K_0$-theory, we need some preliminary definitions and remarks (according to ref. [9] and chapter 4 in ref. [7]). Let $\mathcal{A}$ be a Banach algebra, and let $n \in \mathbb{N}$. We denote the vector space of $(n \times n)$ matrices over $\mathcal{A}$ by $M_n(\mathcal{A})$. An algebraic homomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ induces an algebraic homomorphism $\varphi_n : M_n(\mathcal{A}) \rightarrow M_n(\mathcal{B})$ by the definition $\varphi((a_{ij}))_{i,j=1}^n = (\varphi(a_{ij}))_{i,j=1}^n$.

Suppose that $\mathcal{A}$ is unital. We write $I$ for the identity of $\mathcal{A}$ and denote the group of invertible elements in $\mathcal{A}$ by $G(\mathcal{A})$. The identity of $M_n(\mathcal{A})$ is denoted by $I^{(n)}$, and we let $G_n(\mathcal{A}) := G(M_n(\mathcal{A}))$, the group of invertible elements in $M_n(\mathcal{A})$. For a non-unital algebra $\mathcal{A}$, we denote $\mathcal{A}$ with an identity adjoined by $\mathcal{A}^\ast$. Suppose that $\mathcal{B}$ is a unital algebra containing $\mathcal{A}$ as a subalgebra. Then we identify $\mathcal{A}^\ast = \mathcal{A} + CI$, a subalgebra of $\mathcal{B}$.

Let $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ be Banach spaces. There is a standard one-to-one correspondence between operators $T \in B(\bigoplus_{i=1}^n X_i, \bigoplus_{i=1}^n Y_i)$ and $(n \times n)$-matrices $(T_{ij})_{i,j=1}^n$ with $T_{ij} \in B(Y_j, X_i)$. For every ideal of operators $A$, $T \in A(\bigoplus_{i=1}^n X_i, \bigoplus_{i=1}^n Y_i)$ if and only if $T_{ij} \in A(X_j, Y_i)$ for all $i, j$. In particular, $B(X^n) = M_n(B(X))$, $A(X^n) = M_n(A(X))$.

We define $P_1(\mathcal{A}) := \{ P \in \mathcal{A} : P^2 = P \}$, the set of idempotents in $\mathcal{A}$.

The set of idempotents in $M_n(\mathcal{A})$, $P_n(\mathcal{A}) := P_1(M_n(\mathcal{A}))$, $P_\infty(\mathcal{A}) := \cup_{n \in \mathbb{N}} P_n(\mathcal{A})$.

In $P_\infty(\mathcal{A})$, two idempotents $P, Q \in P_n(\mathcal{A})$ are (algebraically) equivalent if there exist $A, B \in M_n(\mathcal{A})$ such that $P = AB$ and $Q = BA$, denoted by $P \sim_\mathcal{A} Q$.

In $P_\infty(\mathcal{A})$, two idempotents $P \in P_n(\mathcal{A})$ and $Q \in P_m(\mathcal{A})$ are equivalent if their
zero-extensions are equivalent: there exists \( k \geq m, n \), such that
\[
(P \oplus 0_k \oplus 0_{k-m}) \sim_{\alpha} (Q \oplus 0_k \oplus 0_{k-m}), \text{ denoted by } P \sim Q.
\]

Clearly, \( \sim \) is an equivalence relation on \( P_{\omega}(A) \). After properly defining the addition,

\[
V(A) := P_{\omega}(A)/\sim, \text{ } V(A) \text{ is a semigroup.}
\]

Now suppose that algebra \( A \) is unital. Then we define \( K_0(A) \) to be the Grothendieck group of \( V(A) \) (ref. [7]). For \( [P]_V \in V(A) \), we denote the canonical image of \( [P]_V \) in \( K_0(A) \) by \( [P]_0 \), the canonical homomorphism \( \pi : V(A) \rightarrow K_0(A), \pi[P]_V := [P]_0 \in K_0(A) \). The following relations will often be used: for \( P, Q \in P_{\omega}(A) \)
\[
[P]_0 = [Q]_0 \text{ if and only if } \begin{pmatrix} P & 0 \\ 0 & I^{(k)} \end{pmatrix} \sim \begin{pmatrix} Q & 0 \\ 0 & I^{(k)} \end{pmatrix} \text{, for some } k \in \mathbb{N}; \tag{4.1}
\]
and we have the following standard picture of \( K_0(A) \):
\[
K_0(A) = \{ [P]_0 - [Q]_0 : P, Q \in P_{\omega}(A) \}. \tag{4.2}
\]
For each \( n \in \mathbb{N}, P \in P_n(A) \), the identity
\[
[P]_0 + [I^{(n)}]_0 - [P]_0 = [I^{(n)}]_0. \tag{4.3}
\]

In the non-unital case, we define \( K_0(A) \) as a subgroup of \( K_0(A^+) \) in the following way. Let \( s : A^+ \rightarrow A^+ \) denote the scalar map given by \( s(P + \lambda I) := \lambda I(P \in A, \lambda \in \mathbb{C}) \). This is clearly an algebraic homomorphism, and we define
\[
K_0(A) := \{ [P]_0 - [s_n(P)]_0 : n \in \mathbb{N}, P \in P_n(A^+) \}. \tag{4.4}
\]

We now discuss the \( K_0 \)-group of operator ideal. We shall work with complex Banach space throughout this paper. This is because of spectral theory. The following lemmas indicate that in the case of \( A = B(X) \), the equivalence relation \( \sim \) has a nice standard characterization.

**Lemma 6[8,9].** For \( P, Q \in P_{\omega}(A) \), \( A = B(X) \), \( P \sim Q \) if and only if \( \text{Im} P \) is isomorphic to \( \text{Im} Q \).

**Lemma 7.** Let \( X \) be an infinite-dimensional complex Banach space, \( A(X) \) be an ideal of operators contained in the class of Riesz operators \( R(X) \) and \( P \in P(A(X)^+) \). Then \( P \in (F(X)^+) \). \( F(X) \) is the ideal of finite-rank operators contained in \( B(X) \).

**Proof.** Let \( P = S + \lambda I_X \in P_1(A(X)^+) \), \( S \in A(X) \) and \( \lambda \in \mathbb{C} \). Then clearly
\[
P^2 = P \text{ and } S^2 + 2\lambda S - S = [S + (2\lambda - 1)I_X]S = (\lambda - \lambda^2)I_X. \tag{4.5}
\]

Since \( A(X) \) is a proper ideal in \( B(X) \), then \( (\lambda - \lambda^2) = \lambda(1 - \lambda) = 0 \). When \( \lambda = 0 \), \( S^2 = S \) from formula (4.5). So \( S \) is a projection in \( A(X) \), and \( S \in F(X) \). Otherwise, suppose the range of \( S \), \( \text{Im}(S) \), is infinite-dimensional, then \((I_X - S)|_{\text{Im}(S)} = 0 \). This is a contradiction to the fact that \( S \in A(X) \subset R(X) \), \( \dim \text{Ker}(I_X - S) < \infty \). When \( \lambda = 1 \), \( (-S)^2 = -S \) from formula (4.5). Similarly, we have \(-S \in F(X) \) and \( S \in F(X) \). Whenever \( \lambda = 0 \) or \( \lambda = 1 \), we have \( P = S + \lambda I_X \in (F(X)^+) \).

Copyright by Science in China Press 2004
Lemma 8. Let $X$ be an infinite-dimensional complex Banach space, $A$ be an ideal of operators contained in the class of Riesz operators $R(X)$ and $P \in P(A(X)^+)$, then there are idempotents $P_1, \ldots, P_n \in F(X)^+$ and an operator $U \in G_n(A(X)^+)$ satisfying 
\[ \text{Im}(UP) = \text{Im}(P_1 \oplus \cdots \oplus P_n). \]
In particular, the equivalence $P \sim P_1 \oplus \cdots \oplus P_n$ holds in $P_\infty(A(X)^+)$. 

Proof. Let 
\[ P = \begin{pmatrix}
S_{11} + \lambda_{11}I & S_{12} + \lambda_{12}I & \cdots & S_{1n} + \lambda_{1n}I \\
\vdots & \vdots & \ddots & \vdots \\
S_{n1} + \lambda_{n1}I & S_{n2} + \lambda_{n2}I & \cdots & S_{nn} + \lambda_{nn}I
\end{pmatrix}
= (S_{ij})_{n \times n} + (\lambda_{ij}I)_{n \times n}, \]
where $S_{ij} \in A(X)$, $\lambda_{ij} \in \mathbb{C}$. From $P^2 = P$ and being analogous to Lemma 7, we have 
\[ (\lambda_{ij})_{n \times n} \in P_n(\mathbb{C}). \]
Since $(\lambda_{ij})_{n \times n}$ is a $(n \times n)$-projective matrix of complex numbers, there exists an invertible $(n \times n)$-matrix $T = (t_{ij})_{n \times n}$ such that 
\[ T(\lambda_{ij})_{n \times n} T^{-1} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}_{n \times n}. \]
Obviously, there exists $T_0 := (t_{ij}I_X)_{n \times n}$ such that
\[ T_0(\lambda_{ij}I_X)_{n \times n} T_0^{-1} = \begin{pmatrix} I_X & 0 \\ \cdots & I_X \\ 0 & \cdots & 0_{n-k} \end{pmatrix}_{n \times n}, \quad (1 \leq k \leq n). \]
Now, we have 
\[ T_0PT_0^{-1} = T_0(S_{ij})_{n \times n} T_0^{-1} + T_0(\lambda_{ij}I_X)_{n \times n} T_0^{-1} \]
\[ = \begin{pmatrix}
S_{11}^* + I_X & S_{12}^* & S_{13}^* & \cdots & S_{1n}^* \\
S_{21}^* & S_{22}^* + I_X & S_{23}^* & \cdots & S_{2n}^* \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
S_{n1}^* & S_{n2}^* & S_{n3}^* & \cdots & S_{nn}^*
\end{pmatrix} \in P((A(X)^+)^n). \]

According to Corollary 7 of Proposition 11, there exists an isomorphism $U \in G_n(A(X)^+)$ and a series of projections $P_i \in A(X)^+$ ($i = 1, \ldots, n$) such that 
\[ \text{Im}(UT_0PT_0^{-1}) = \text{Im}(P_1 \oplus P_2 \oplus \cdots \oplus P_n) \]
and hence $\text{Im}(U_0P) = \text{Im}(P_1 \oplus P_2 \oplus \cdots \oplus P_n)$, $U_0 = UT_0$.

By Lemma 7, we have $P_i \in F(X), i = 1, \ldots, n$.

According to the definition of equivalence relation in $P_\infty(A(X)^+) = \bigcup_{n \in \mathbb{N}} P_n(A(X)^+)$, we have proved that $P \sim (P_1 \oplus P_2 \oplus \cdots \oplus P_n)$.

The following lemma is a generalization of Lemma 3.5 in ref. [9].

Lemma 9. Suppose that $A(X) \subset R(X)$, $Q \in P_\infty(F(X))$ is an operator of rank one and $n \in \mathbb{N}$. For every $P \in P_n(F(X)), Q \in P_n(F(X))$, the formulæ 
\[ [P]_0 = (r_kP)[Q]_0 \quad \text{and} \quad [I^{(n)}]_0 - [P]_0 - [I^{(n)}]_0 = -(r_kP)[Q]_0 \]

www.scichina.com
hold in $K_0(A(X)^+)$). In particular, $[P]_0 - [s_n(P)]_0 \in \mathbb{Z}[Q]_0$ for every $P \in P_1(M_n(F(X)^+))$ ($rKP$ denotes the rank of $P$).

**Proof.** Suppose $P \in P_n(F(X))$. By Lemma 8, we have $P \sim (P_1 \oplus P_2 \oplus \cdots \oplus P_n)$, then $rKP = \sum_{i=1}^n rKP_i < \infty$. By Lemma 6, for every $i, P_i \sim Q \oplus \cdots \oplus Q$, then

$$[P]_0 = \sum_{i=1}^n [P_i]_0 = \sum_{i=1}^n (rKP_i)[Q]_0 = (\sum_{i=1}^n (rKP_i))[Q]_0 = (rKP)[Q]_0.$$  \hspace{1cm} (4.6)

In addition, by formulae (4.3) and (4.6), we have $[I^{(n)} - P]_0 - [I^{(n)}]_0 = - (rKP)[Q]_0$.

For any $P \in P_\infty(A(X)^+)$, there exists some $n$ such that $P \in P_n(A(X)^+) = P_1(M_n(A(X)^+))$. Since $P \sim P_1 \oplus P_2 \oplus \cdots \oplus P_n$, $P_i \in F(X)^+$ and by Lemma 7, $P_i = S_i + \lambda_i I^{(1)}$, $S_i \in F(X)$, $\lambda_i = 0$ or $1$, $i = 1, 2, \cdots, n$. There is no loss of generality to suppose that

$$\begin{cases}
P_i = S_i + I^{(1)}, \\
1 \leq i \leq k, \text{ by formula (4.6), } [I^{(1)} - P_i]_0 = rK(I^{(1)} - P_i)[Q]_0, \\
P_i = S_i + 0I^{(1)} = S_i, \\
k < i \leq n, \text{ according to formula (4.6), } [P_i]_0 = (rKP_i)[Q]_0.
\end{cases}$$

Then we have

$$[P]_0 = ([P_1]_0 + [P_2]_0 + \cdots [P_k]_0) + ([P_{k+1}]_0 + \cdots + [P_n]_0)
= [s_k(P_1 \oplus \cdots \oplus P_k)]_0 - (I^{(1)} - P_1) \oplus (I^{(1)} - P_2) \oplus \cdots \oplus (I^{(1)} - P_k)]_0
+ rK(P_{k+1} \oplus \cdots \oplus P_n)[Q]_0.$$  

So we have

$$[P]_0 - [s_n(P)]_0 = [P]_0 - [s_k(P_1 \oplus \cdots \oplus P_k)]_0
= - rK((I^{(1)} - P_1) \oplus \cdots \oplus (I^{(1)} - P_k))[Q]_0
+ rK(P_{k+1} \oplus \cdots \oplus P_n)[Q]_0
= \{rK(P_{k+1} \oplus \cdots \oplus P_n) - rK((I^{(1)} - P_1) \oplus \cdots \oplus (I^{(1)} - P_k))[Q]_0 \in \mathbb{Z}[Q]_0.$$  

To complete the proof of $K_0(A(X)) \approx \mathbb{Z}$, we need a fact that the abelian semigroup $P_\infty(A(X)^+)/\sim$ has cancellation. It can be proved by replacing $S$ with $A$ in some related results (that is, Proposition 3.1 and Proposition 3.8 in ref. [9]). We prove it in some details.

**Lemma 10.** Let $X$ be an infinite-dimensional complex Banach space, $A(X)$ is a non-zero, closed ideal of operators contained in the class of Riesz operators. Let $n \in \mathbb{N}$, and $T = (T_{ij})_{i,j=1}^n \in M_n(A(X)^+)$. The spectrum of $T$ is independent of the question of whether it is calculated in $M_n(A(X)^+)$ or $M_n(B(X))$, i.e.

$$\sigma_{M_n(A(X)^+)}(T) = \sigma_{M_n(B(X))}(T),$$

and it is countable.

Copyright by Science in China Press 2004
Proof. Recall that \( A(X)^+ := A(X) + \mathbb{C}I_X \), it is a closed subalgebra of \( B(X) \) and identity \( I_X \in B(X) \). Write \( T_{ij} = S_{ij} + \lambda_{ij}I_X \), where \( S_{ij} \in A(X) \), \( \lambda_{ij} \in \mathbb{C} \). We can also write \( T = \sum_{i,j=1}^{n} T_{ij} = \sum_{i,j=1}^{n} (S_{ij} + \lambda_{ij}) = S + A \), where \( S = \sum_{i,j=1}^{n} S_{ij} \in M_n(A(X)) \subset A(X^n) \), \( A = \sum_{i,j=1}^{n} \lambda_{ij}I_{ij} \). Since \( A(X) \) is called the perturbation class associated with the set of Fredholm operators (see Proposition 10 (b)), it follows from the definition of (Wolf) essential spectrum that

\[
\sigma_e(T) = \sigma_e(A) \subseteq \sigma_{B(X^n)}(A) = \sigma_{M_n(B(X))}(A) \subseteq \sigma_{M_n(\mathbb{C})}(A).
\]

In particular, \( T \) is an element of \( M_n(B(X)) = B(X^n) \), its essential spectrum \( \sigma_e(T) = \sigma_e(A) \subseteq \sigma_{M_n(\mathbb{C})}(A) \) is finite. \( \sigma_k(T) \) (the Kato spectrum of \( T \)) is countable. By Theorem 5.33 in ref. [18] § IV, the finiteness of \( \sigma_k(T) \) implies the countability of \( \sigma(T) \). Lastly, we notice that \( M_n(A(X)^+) \) is a closed subalgebra of \( M_n(B(X)) \), and they have the same identity \( I_X^n \). By the corollary of Theorem 10.18 in ref. [21], \( \sigma_{M_n(A(X)^+)}(T) = \sigma_{M_n(B(X))}(T) \).

**Lemma 11.** The semigroup \( P_\infty(A(X)^+)/\sim \) has additive cancellation. That is, when \( P_1, P_2, Q \in P_\infty(A(X)^+) \) satisfy

\[
\begin{pmatrix} P_1 & 0 \\ 0 & Q \end{pmatrix} \sim \begin{pmatrix} P_2 & 0 \\ 0 & Q \end{pmatrix} \equiv [P_1]_V + [Q]_V = [P_2]_V + [Q]_V,
\]

then \( P_1 \sim P_2 \) or \( [P_1]_V = [P_2]_V \).

**Proof.** Since \( A(X)^+ \) is a unital Banach algebra, for every \( x \in A(X)^+ \), \( \sigma(x) \) is countable by Lemma 10. By Proposition 3.6 in ref. [9], \( A(X)^+ \) has stable rank one, that is, the group of invertible elements \( G(A(X)^+) \) is dense in \( A(X)^+ \). By Proposition 3.7 in ref. [9], for every \( n \in \mathbb{N} \), \( M_n(A(X)^+) \) has stable rank one. Finally, \( P_\infty(A(X)^+)/\sim \) has cancellation by Proposition 3.8 in ref. [9].

Till now, completing the proof of \( K_0(A(X)) = \mathbb{Z} \) is easy. Suppose projection \( Q \in P_\infty(B(X)) \) is of rank one, we have to show that

\[
\omega : v \mapsto v[Q]_0, \quad \mathbb{Z} \to K_0(A(X))
\]

is a group isomorphism. Clearly, \( \omega \) is additive. To show that \( \omega \) is surjective, let \( g \in K_0(A(X)) \) be given. By formula (4.4): \( K_0(A(X)) = \{ [P]_0 - [s_n(P)]_0 \mid n \in \mathbb{N}, P \in P_n(A(X)^+) \} \), we can take \( n \in \mathbb{N} \) and \( P \in P_n(A(X)^+) \) such that \( g = [P]_0 - [s_n(P)]_0 \). By Lemma 8, we can take \( P_1, \ldots, P_n \in P(F(X)^+) \) for which \( P \sim P_1 \oplus \cdots \oplus P_n \). Consequently

\[
g = [P_1 \oplus \cdots \oplus P_n]_0 - [s_n(P_1 \oplus \cdots \oplus P_n)]_0 = \sum_{i=1}^{n} ([P_i]_0 - [s(P_i)]_0) \in \mathbb{Z}[Q]_0
\]

by Lemma 9.

Next we shall show that \( \omega \) is injective. Otherwise, there is a natural number \( n_0 \) satisfying

\[
[0]_0 = n_0[Q]_0 = [Q \oplus \cdots \oplus Q]_0 \in K_0(A(X)) \subset K_0(A(X)^+).
\]

Then, by Lemma 11, we should have \([Q]_0 = [0]_0\). This is in contradiction with the fact that \( Q \neq 0 \).
We now proceed to considering group $K_1$. We shall give some definitions and remarks (see refs. [8,9] or ch 4 in ref. [7], ch 8 in ref. [6]). Suppose that $\mathcal{A}$ is a unital Banach algebra, and define $G_\infty(\mathcal{A}) := \bigcup_{n \in \mathbb{N}} G_n(\mathcal{A})$, where $G_n(\mathcal{A})$ is the group of invertible elements in $M_n(\mathcal{A})$. For $A \in G_m(\mathcal{A})$ and $B \in G_n(\mathcal{A})$, we say that $A \sim_1 B$ provided that there is a natural number $k \geq \max(m, n)$ and a continuous path $t \mapsto W_t, [0, 1] \rightarrow G_k(\mathcal{A})$,

$$W_0 = \begin{pmatrix} A & 0 \\ 0 & I_{(k-m)} \end{pmatrix} \quad \text{and} \quad W_1 = \begin{pmatrix} B & 0 \\ 0 & I_{(k-n)} \end{pmatrix}. $$

Clearly, $\sim_1$ is an equivalence relation on $G_\infty(\mathcal{A})$. It is easy to check that the operation

$$([A]_1, [B]_1) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, K_1(\mathcal{A}) \times K_1(\mathcal{A}) \rightarrow K_1(\mathcal{A}) $$

is well-defined and turns $K_1(\mathcal{A})$ into a commutative group.

For a non-unital Banach algebra $\mathcal{A}$, we define $K_1(\mathcal{A}) := K_1(\mathcal{A}^+)$. Nowadays we can easily prove that $K_1(A(X)^+) = 0$. For every $T \in A(X) \subset \mathbb{R}(X)$, each element $T + \lambda I \in A(X)^+$ has a countable spectrum. By Proposition 3.2 in ref. [9], each $T \in G(A(X)^+)$ belongs to the main component $G^0(A(X)^+)$, the connected component which contains identity $I_X$. Hence we have $K_1(A(X)) = K_1(A(X)^+) = 0$.

Until now we have completed the proof of Theorem 3.

**Corollary 9.** Let $X$ be an infinite-dimensional complex Banach space. If $A$ is an ideal in $B(X)$ satisfying one of the following conditions, then $K_0(A) = \mathbb{Z}$ and $K_1(A) = 0$: (a) $A = \overline{F}(X)$, the closure of finite-rank operators; (b) $A = K(X)$, the ideal of compact operators; (c) $A = S(X)$, the ideal of strictly singular operators; (d) $A = S^c(X)$, the ideal of strictly cosingular operators; (e) $A = J(X)$, the ideal of radical operators.

5 **K-group of operator algebras on some G-M-type spaces**

As an application of Theorem 3, we can easily compute the $K$-group of Banach algebra $B(X)$ on some G-M-type spaces with operator structure $B(X) = A(X) + CI_X$.

**Proposition 12.** Let $X$ be an infinite-dimensional complex Banach space, $B(X) = \{\lambda I_X + T : \lambda \in \mathbb{C}, T \in A(X)\}$, where $A(X)$ is a closed ideal of operators contained in the class of Riesz operators. Then $K_0(B(X)) = \mathbb{Z}^2$ and $K_1(B(X)) = 0$.

**Proof.** First, under the condition that $B(X) = A(X) + CI_X = A(X)^+$, we can get $K_1(B(X)) = K_1(A(X)^+) = K_1(A(X)) = 0$ following the definition of $K_1$-group and Theorem 3.

Secondly, from the short exact sequence

$$0 \rightarrow A(X) \rightarrow B(X) \rightarrow B(X)/A(X) \rightarrow 0, \quad (5.1)$$

Copyright by Science in China Press 2004
we can obtain a cyclic six-term exact sequence
\[
K_0(A(X)) \rightarrow K_0(B(X)) \rightarrow K_0(B(X)/A(X)) \\
\uparrow \\
K_1(B(X)/A(X)) \leftarrow K_1(B(X)) \leftarrow K_1(A(X)). \quad (5.2)
\]

Under the condition of the proposition, we know that \(K_0(A(X)) = \mathbb{Z}\) and \(K_1(A(X)) = 0\) from Theorem 3. Furthermore, it is well-known that: \(K_0(B(X)/A(X)) = K_0(\mathbb{C}) = \mathbb{Z}\) and \(K_1(B(X)/A(X)) = K_1(\mathbb{C}) = 0\). By the exactness, we have \(K_0(B(X)) = K_0(A(X)) \oplus K_0(B(X)/A(X)) = \mathbb{Z}\) and \(K_1(B(X)) = K_1(B(X)/A(X)) = 0\).

**Proof 2.** Under the condition of Proposition 12, short exact sequence formula (5.1) is obviously (strongly) split exact. Because functors \(K_0\) and \(K_1\) preserve (strongly) short exact sequences (see refs. [6,9]), we can easily get the answer without using formula (5.2).  

**Corollary 10** [9]. For any complex \(H.I.\) space \(X, K_0(B(X)) = \mathbb{Z}\) and \(K_1(B(X)) = 0\).

**Corollary 11.** For any complex \(Q.H.I\_C.\) space \(X, K_0(B(X)) = \mathbb{Z}\) and \(K_1(B(X)) = 0\).

**Corollary 12.** For any complex \(Q.I\_C.\) space \(X, K_0(B(X)) = \mathbb{Z}\) and \(K_1(B(X)) = 0\).

**Corollary 13.** For any complex \(Q.I.\) space \(X, K_0(B(X)) = \mathbb{Z}\) and \(K_1(B(X)) = 0\).

Additionally, there still is an open problem (the so-called Pisier’s problem) in the structural theory of Banach spaces after the appearance of the series of results by Gowers and Maurey, namely, is there an infinite-dimensional Banach space \(X\) such that \(B(X) = C(I_X + K(X))\)? It has drawn great attention from scientists in this field (see refs. [4,14]). Now, we know that if it exists, then the K-groups of its operator algebras can be computed.

**Acknowledgements** The main part of this work was done during the author’s visit to the Morningside Center of Mathematics (MCM), Chinese Academy of Sciences. The authors wish to thank Prof. Li for his invitation and suggestions. The authors also thank the members of MCM for their hospitality. The first author is grateful to his teacher Lin Chen for helpful correspondence and conversations about K-theory. This work was supported by the National Natural Science Foundation of China (Grant No. 10171014) and the Natural Science Foundation of Fujian Province of China.

**References**


www.scichina.com