Our topic will be to consider the “small” subspaces of $L_p = L_p[0, 1]$ for $2 < p < \infty$ with an objective of classifying when $X \subseteq L_p$ contains or embeds isomorphically into one of these spaces.

We assume always $X \subseteq L_p$ is infinite dimensional. $X \hookrightarrow Y$ means $X$ is isomorphic to a subspace of $Y$. $X \overset{K}{\sim} Y$ means $X$ is $K$-isomorphic to $Y$.

We recall some of the basic structure of $L_p$. The Haar basis $(h_i)$ is a monotone basis for $L_p$ ($1 \leq p < \infty$) which is $K_p$-unconditional iff $1 < p < \infty$;

$$\left\| \sum a_i h_i \right\| \leq K_p \left\| \sum \pm a_i h_i \right\|$$

for all signs $\pm$ and all reals $(a_i)$. 
\( L_p \) contains the following “small” subspaces

- \( \ell_p \) (isometrically): If \((x_i) \subseteq S_{L_p}\) are disjointly supported then

\[
\left\| \sum a_i x_i \right\| = \left( \int \left| \sum a_i x_i(t) \right|^p dt \right)^{1/p} \]
\[
= \left( \sum \int \left| a_i \right|^p \left| x_i(t) \right|^p dt \right)^{1/p} \]
\[
= \left( \sum \left| a_i \right|^p \right)^{1/p}. \]

- \( \ell_2 \) (isomorphically) via the Rademacher functions \((r_n)\). \((r_n)\) are \(\pm 1\) valued independent random variables of mean 0.
Khintchin’s inequality: For $2 < p < \infty$,

$$
\left( \sum |a_n|^2 \right)^{1/2} = \left\| \sum a_n r_n \right\|_2 \leq \left\| \sum a_n r_n \right\|_p \\
\leq B_p \left( \sum |a_n|^2 \right)^{1/2}.
$$

- $\ell_2$ (isometrically) via a sequence of symmetric Gaussian independent random variables in $S_{L_p}$
- $(\ell_2 \oplus \ell_p)_p$ (isometrically)
- $(\sum \ell_2)_p \equiv \{(x_i) : x_i \in \ell_2 \text{ for all } i \}
\text{ and } \| (x_i) \| = \left( \sum \| x_i \|_2^p \right)^{1/p} < \infty \} \text{ (isometrically)}$

So again, our topic will be to characterize when $X \subseteq L_p$, $2 < p < \infty$, embeds isomorphically into or contains isomorphically one of the four spaces $\ell_p$, $\ell_2$, $\ell_p \oplus \ell_2$ or $(\sum \ell_2)_p$. We begin with some known results.

**Proposition 1.** Let $2 < p < \infty$ and let $(x_i) \subseteq S_{L_p}$ be $\lambda$-unconditional. Then for all $(a_n) \subseteq \mathbb{R}$,

$$
\lambda^{-1} \left( \sum |a_n|^p \right)^{1/p} \leq \left\| \sum a_n x_n \right\|_p \leq \lambda B_p \left( \sum |a_n|^2 \right)^{1/2}.
$$
Proof. For \( t \in [0, 1] \),
\[
\| \sum a_n x_n \|_p \leq \lambda \| \sum a_n x_n r_n(t) \|_p
\]
and so
\[
\| \sum a_n x_n \|_p^p \leq \lambda^p \int_0^1 \| \sum a_n x_n r_n(t) \|_p^p \, dt
\]
(Fubini)
\[
= \lambda^p \int_0^1 \int_0^1 \left| \sum a_n x_n(s) r_n(t) \right|^p \, dt \, ds
\]
\[
\leq (\lambda B_p)^p \int_0^1 \left( \sum a_n^2 x_n(s)^2 \right)^{p/2} \, ds
\]
\[
\leq (\lambda B_p)^p \left( \sum \| a_n^2 x_n^2 \|_{p/2} \right)^{p/2}
\]
(by the triangle inequality in \( L_{p/2} \))
\[
= (\lambda B_p)^p \left( \sum |a_n|^2 \right)^{p/2}.
\]
This gives the upper \( \ell_2 \)-estimate.

Similarly,
\[
\lambda^p \left\| \sum a_n x_n \right\|_p^p \geq \int_0^1 \left( \sum a_n^2 x_n^2(s) \right)^{p/2} \, ds
\]
\[
\geq \int_0^1 \sum |a_n|^p |x_n(s)|^p \, ds = \sum |a_n|^p
\]
(using \( \| \cdot \|_p \leq \| \cdot \|_{\ell_2} \)).
So unconditional basic sequences in $L_p$ are trapped between the $\ell_p$ and $\ell_2$ norms.

**Theorem 2** (Kadets and Pełczyński, 1961/62). Let $X \subseteq L_p$, $2 < p < \infty$. Then $X \sim \ell_2$ iff $\| \cdot \|_2 \sim \| \cdot \|_p$ on $X$; i.e., for some $C$, $\| x \|_2 \leq \| x \|_p \leq C \| x \|_2$ for all $x \in X$. Moreover there is a projection $P : L_p \to X$.

Note first that if $x \in S_{L_p}$ and $m\{ t : |x(t)| \geq \varepsilon \} \geq \varepsilon$ then $\| x \|_2 \leq \| x \|_p \leq \varepsilon^{-3/2} \| x \|_2$. Indeed

$$\int |x(t)|^2 \, dt \geq \int_{[|x| \geq \varepsilon]} |x(t)|^2 \, dt \geq \varepsilon^3.$$  

The direction requiring proof is if $X \sim \ell_2$ then $\| \cdot \|_2 \sim \| \cdot \|_p$ on $X$. If not we can find $(x_i) \subseteq S_X$, $x_i \overset{w}{\to} 0$, so that for all $\varepsilon > 0$, $\lim_n m[|x_n| \geq \varepsilon] = 0$, where $m$ is Lebesgue measure on $[0, 1]$. From this we can construct a subsequence $(x_{n_i})$ and disjointly supported $(f_i) \subseteq S_{L_p}$ with $\lim_i \| x_{n_i} - f_i \| = 0$. Hence by a perturbation argument a subsequence of $(x_i)$ is equivalent to the unit vector basis of $\ell_p$ which contradicts $X \sim \ell_2$. 
The projection onto $X$ with $\|x\|_p \leq C\|x\|_2$ for $x \in X$ is given by the orthogonal projection $P : L_2 \to X$ acting on $L_p$. For $y \in L_p$,

$$\|Py\|_p \leq C\|Py\|_2 \leq C\|y\|_2 \leq C\|y\|_p .$$

□

**Remarks.** The proof yields that if $X \subseteq L_p$, $2 < p < \infty$, and $X \not\sim \ell_2$ then for all $\varepsilon > 0$, $\ell_p^{1+\varepsilon} \hookrightarrow X$.

Pełczyński and Rosenthal (1974/75) proved that if $X \sim \ell_2$ then $X$ is $C(K)$-complemented in $L_p$ via a change of density argument. Maurey (1974) obtained $C(K)$ is a linear function of $K$.

**Theorem 3.** [Johnson and Odell, 1974] Let $2 < p < \infty$, $X \subseteq L_p$. Then $X \hookrightarrow \ell_p \iff \ell_2 \not\hookrightarrow X$. ([Kalton and Werner, 1995] If $\ell_2 \not\hookrightarrow X$ then for all $\varepsilon > 0$, $X \hookrightarrow \ell_p^{1+\varepsilon}$.)

The scheme of the argument is to show if $\ell_2 \not\hookrightarrow X$ then there is a blocking $(H_n)$ of the Haar basis into an FDD so that $X \hookrightarrow (\sum H_n)_p$ in a natural way; $x = \sum x_n$, $x_n \in H_n \to (x_n) \in (\sum H_n)_p$. By Pełczyński (1960), $(\sum H_n)_p \sim \ell_p$ since it is complemented in $\ell_p$. 
Let’s digress for a moment to discuss subspaces of $L_p$ ($1 < p < 2$). The situation is more complemented here: $L_q \hookrightarrow L_p$ if $p \leq q \leq 2$. Johnson characterized when $X \subseteq L_p$ ($1 < p < 2$) embeds into $\ell_p$.

**Theorem 4.** [Johnson, 1977] Let $X \subseteq L_p$, $1 < p < 2$. Then $X \hookrightarrow \ell_p$ if there exists $K < \infty$ so that for all weakly null $(x_i) \subseteq S_X$ some subsequence is $K$-equivalent to the unit vector basis of $\ell_p$.

These results were unified using the infinite asymptotic game/weakly null trees machinery.

**Theorem 5.** Let $X \subseteq L_p$, $1 < p < \infty$. Then $X \hookrightarrow \ell_p$ iff every weakly null tree in $S_X$ admits a branch equivalent to the unit vector basis of $\ell_p$.

A weakly null tree in $S_X$ is $(x_\alpha)_{\alpha \in T_\infty} \subseteq S_X$ where $T_\infty = \{(n_1, \ldots, n_k) : k \in \mathbb{N}, n_1 < \cdots < n_k \text{ are in } \mathbb{N}\}$. A node in $T_\infty$ is all $(x_{(\alpha,n)})_{n>n_k}$ where $\alpha = (n_1, \ldots, n_k)$ or $\alpha = \emptyset$. The tree is weakly null means each node is a weakly null sequence. A branch is $(x_i)$ given by $x_i = x_{(n_1, \ldots, n_i)}$ for some subsequence $(n_i)$ of $\mathbb{N}$. 
The proof is the same as that in [Odell and Th. Schlumprecht, 2002] used to characterize when a reflexive space $X \hookrightarrow (\sum F_n)_p$. It yields a blocking $(H_n)$ of $(h_n)$ so that $X \hookrightarrow (\sum H_n)_p$, naturally.

We won’t go into the details here. The idea is to first produce $C$ and a blocking $(G_n)$ of $(h_i)$ and $\bar{\delta} = (\delta_i)$, $\delta_i \downarrow 0$, so that $\bar{\delta}$-skipped block sequences $(x_n) \subseteq S_X$ are all $C$-equivalent to the unit vector basis of $\ell_p$. Then to form a further blocking $(H_n)$ of $(G_n)$ using a decomposition lemma of Johnson’s which allows us to essentially decompose any $x \in S_X$ into a linear combination of such a $\bar{\delta}$-skipped block sequence.

Returning to $X \subseteq L_p$ $(2 < p < \infty)$ we have seen that one of these holds:

- $X \sim \ell_2$
- $X \hookrightarrow \ell_p$
- $\ell_p \oplus \ell_2 \hookrightarrow X$

Our goal will be to characterize when $X \hookrightarrow \ell_p \oplus \ell_2$ and if not to then show that $(\sum \ell_2)_p \hookrightarrow X$.

First we recall one more old result.
**Theorem 6.** [Johnson and Odell, 1981] Let $X \subseteq L_p$, $2 < p < \infty$. Assume there exists $Y \subseteq \ell_p \oplus \ell_2$ and a quotient (onto) map $Q : Y \to X$. Then $X \hookrightarrow \ell_p \oplus \ell_2$.

This is an answer, of a sort, to when $X \hookrightarrow \ell_p \oplus \ell_2$ but it is not an intrinsic characterization. The proof however provides a clue as to how to find one. The isomorphism $X \hookrightarrow \ell_p \oplus \ell_2$ is given by a blocking $(H_n)$ of $(h_i)$ so that $X$ naturally embeds into

\[
\left( \sum H_n \right)_p \oplus \left( \sum (H_n, \| \cdot \|_2) \right)_2 \sim \ell_p \oplus \ell_2.
\]

Before proceeding we recall some more inequalities.

**Theorem 7.** [Rosenthal, 1970] Let $2 < p < \infty$. There exists $K_p < \infty$ so that if $(x_i)$ is a normalized mean zero sequence of independent random variables in $L_p$ then for all $(a_i) \subseteq \mathbb{R},$

\[
\left\| \sum a_i x_i \right\|_p \overset{K_p}{\sim} \left( \sum |a_i|^p \right)^{1/p} \vee \left( \sum |a_i|^2 \|x_i\|_2^2 \right)^{1/2}.
\]

Note that in this case $[(x_i)] \hookrightarrow \ell_p \oplus \ell_2$ via the embedding

\[
\sum a_i x_i \longmapsto \left( (a_i)_i, (a_i \|x_i\|_2)_i \right) \in \ell_p \oplus \ell_2.
\]
The next result generalizes this to martingale difference sequences, e.g., block bases of \( (h_i) \).

**Theorem 8.** [Burkholder, 1973; Burkholder, Davis, and Gundy, 1972] Let \( 2 < p < \infty \). There exists \( C_p < \infty \) so that if \( (z_i) \) is a martingale difference sequence in \( L_p \) with respect to the sequence of \( \sigma \)-algebras \( (\mathcal{F}_n) \), then

\[
\left\| \sum z_i \right\|_p \sim C_p \left( \sum \|z_i\|_p^p \right)^{1/p} \vee \left( \sum \mathbb{E}_{\mathcal{F}_i}(z_{i+1}^2) \right)^{1/2}. 
\]

Now suppose that \( (x_i) \subseteq S_{L_p} \) is weakly null. Passing to a subsequence we obtain \( (y_i) \) which, by perturbing, we may assume is a block basis of \( (h_i) \). Passing to a further subsequence we may assume \( \varepsilon \equiv \lim_i \|y_i\|_2 \) exists. If \( \varepsilon = 0 \) a subsequence of \( (y_i) \) is equivalent to the unit vector basis of \( \ell_p \) by the [Kadets and Pełczyński, 1961/62] arguments. Otherwise we have (essentially)

\[
\varepsilon \left( \sum |a_i|^2 \right)^{1/2} = \left\| \sum a_i y_i \right\|_2 \leq \left\| \sum a_i y_i \right\|_p \leq C(p) \left( \sum |a_i|^2 \right)^{1/2},
\]

using the fundamental inequality, Proposition 1.
Johnson, Maurey, Schechtman and Tzafriri obtained a stronger version of this dichotomy using Theorem 8.

**Theorem 9.** [Johnson, Maurey, Schechtman, and Tzafriri, 1979] Let $2 < p < \infty$. There exists $D_p < \infty$ with the following property. Every normalized weakly null sequence in $L_p$ admits a subsequence $(x_i)$ satisfying, for some $w \in [0, 1]$ and all $(a_i) \subseteq \mathbb{R}$,

$$\left\| \sum a_i x_i \right\|_p \sim D_p \left( \sum |a_i|^p \right)^{1/p} \vee w \left( \sum |a_i|^2 \right)^{1/2}.$$

We are now ready for an intrinsic characterization of when $X \subseteq L_p$ embeds into $\ell_p \oplus \ell_2$. 
Theorem 10. [Haydon, Odell, and Schlumprecht] Let $X \subseteq L_p$, $2 < p < \infty$. The following are equivalent.

a) $X \hookrightarrow \ell_p \oplus \ell_2$

b) Every weakly null tree in $S_X$ admits a branch $(x_i)$ satisfying for some $K$ and all $(a_i)$

$$\left\| \sum a_i x_i \right\| \preceq K \left( \sum |a_i|^p \right)^{1/p} \vee \left\| \sum a_i x_i \right\|_2$$

$$\approx \left( \sum |a_i|^p \right)^{1/p} \vee \left( \sum |a_i|^2 \|x_i\|_2^2 \right)^{1/2}$$

c) Every weakly null tree in $S_X$ admits a branch $(x_i)$ satisfying for some $K$ and $(w_i) \subseteq [0, 1]$ and all $(a_i)$,

$$\left\| \sum a_i x_i \right\| \preceq K \left( \sum |a_i|^p \right)^{1/p} \vee \left( \sum |a_i|^2 w_i^2 \right)^{1/2}.$$ 

d) There exists $K$ so that every weakly null sequence in $S_X$ admits a subsequence $(x_i)$ satisfying the condition in b):

$$\left\| \sum a_i x_i \right\| \preceq K \left( \sum |a_i|^p \right)^{1/p} \vee \left\| \sum a_i x_i \right\|_2$$

$$\approx \left( \sum |a_i|^p \right)^{1/p} \vee \left( \sum |a_i|^2 \varepsilon^2 \right)^{1/2}$$

where $\varepsilon = \lim_i \|x_i\|_2$. 

Condition c) just says that every weakly null tree in $S_X$ admits a branch equivalent to a block basis of the natural basis for $\ell_p \oplus \ell_2$.

Conditions b) and c) do not require $K$ to be universal but the “all weakly null trees...” hypothesis yields this.

Condition d) is an anomaly in that usually “every sequence has a subsequence...” is a vastly different condition than “every tree admits a branch...”. Here the special nature of $L_p$ is playing a role.

The embedding of $X$ into $\ell_p \oplus \ell_2$ will follow the clue from Theorem 6 by producing a blocking $(H_n)$ of $(h_i)$ and embedding $X$ naturally into

$$\left( \sum H_n \right)_p \oplus \left( \sum (H_n, \| \cdot \|_2) \right)_2.$$

Now b) $\Rightarrow$ a) by using the infinite asymptotic game machinery of Odell and Schlumprecht (2006). We note the argument proves the following
Theorem 11. Let $X$ and $Y$ be Banach spaces with $X$ reflexive. Let $V$ be a space with a 1-subsymmetric normalized basis $(v_i)$ and let $T : X \rightarrow Y$ be a bounded linear operator. Assume that for some $C$ every normalized weakly null tree in $X$ admits a branch $(x_n)$ satisfying:

$$\left\| \sum a_n x_n \right\|_X \lesssim C \left\| \sum a_n v_n \right\|_V \vee \left\| T\left( \sum a_n x_n \right) \right\|_Y .$$

Then if $X \subseteq Z$, a reflexive space with an FDD($E_i$), there exists a blocking $(G_i)$ of $(E_i)$ so that $X$ naturally embeds into $(\sum G_i)_V \oplus Y$.

This is applied to $V = \ell_p$, $Z = L_p$ and $Y = L_2$ where $T : X \rightarrow L_2$ is the identity map.

So we obtain b) $\Rightarrow$ a) and clearly a) $\Rightarrow$ c). To see c) $\Rightarrow$ b) we begin with a weakly null tree in $S_X$ and choose a branch $(x_i)$ satisfying the c) condition:

$$\left\| \sum a_i x_i \right\| \sim K \left( \sum |a_i|^p \right)^{1/p} \vee \left( \sum |a_i|^2 |w_i|^2 \right)^{1/2} .$$

We want to say that for some $K'$,

$$\left\| \sum a_i x_i \right\| \sim K' \left( \sum |a_i|^p \right)^{1/p} \vee \left\| \sum a_i x_i \right\|_2 .$$
(We have $K'$ by the fundamental inequality.)

If this fails we can find a block basis $(y_n)$ of $(x_n)$,

$$y_n = \sum_{i=k_{n-1}+1}^{k_n} a_i x_i , \text{ with } \sum_{i=k_{n-1}+1}^{k_n} w_i^2 a_i^2 = 1$$

and

$$\left( \sum_{i=k_{n-1}+1}^{k_n} |a_i|^p \right)^{1/p} \lor \|y_n\|_2 < 2^{-n} .$$

But then from the c) condition $(y_n)$ is equivalent to the unit vector basis of $\ell_2$ and from the above condition a subsequence is equivalent to the unit vector basis of $\ell_p$, a contradiction.

Note that b) $\Rightarrow$ d) since if $(x_i)$ is a normalized weakly null sequence and we define $(x_\alpha)_{\alpha \in T_\infty}$ by $x_{(n_1, \ldots, n_k)} = x_{n_k}$ then the branches of $(x_\alpha)_{\alpha \in T_\infty}$ coincide with the subsequences of $(x_n)$. Note that the condition d) just says we may take the weight "w" in [Johnson, Maurey, Schechtman, and Tzafriri, 1979] to be "$\lim_i \|x_i\|_2$". It remains to show d) $\Rightarrow$ b) in Theorem 10. The idea is to use Burkholder's inequality using d) on nodes of a weakly null
tree, following the scheme of [Johnson, Maurey, Schechtman, and Tzafriri, 1979] to accomplish this. That argument will obtain a branch \( (x_n) = (x_{\alpha_n}) \), \( \alpha_n = (m_1, \ldots, m_n) \) with

\[
\left\| \sum a_i x_i \right\| \sim \left( \sum |a_i|^p \right)^{1/p} \vee \left( \sum w_i^2 a_i^2 \right)^{1/2}
\]

where \( w_i \overset{C(p)}{\sim} \lim_n \|x_{(\alpha,n,n)}\|_2 \) using d).

**Theorem 12.** Let \( X \subseteq L_p \), \( 2 < p < \infty \). If \( X \) does not embed into \( \ell_p \oplus \ell_2 \) then \( (\sum \ell_2)_p \hookrightarrow X \).

The idea of the proof is to produce a sequence of “skinny” \( \ell_2 \)'s inside \( X \), provided \( X \not\hookrightarrow \ell_p \oplus \ell_2 \).

**Lemma.** Assume for some \( K \) and all \( n \) there exists \( (x_i^n)_{n=1}^{\infty} \subseteq S_X \) with \( \lim_i \|x_i^n\|_2 = \varepsilon_n \downarrow 0 \) and \( (x_i^n)_i \) is \( K \)-equivalent to the unit vector basis of \( \ell_2 \). Then \( (\sum \ell_2)_p \hookrightarrow X \).

**Sketch of proof.** Note that if \( y = \sum_i a_i x_i^n \) has norm 1 then, assuming as we may that \( (x_i^n)_i \) is a block basis of \( (h_i) \) and \( \|x_i^n\|_2 \approx \varepsilon_n \) then

\[
\|y\|_2 \approx \left( \sum a_i^2 \|x_i^n\|_2^2 \right)^{1/2} \lesssim K \varepsilon_n .
\]
So we have a sequence of skinny $K - \ell_2$’s inside of $X$. We would like to have if $y^n \in [(x^n_i)_i]$ then they are essentially disjointly supported so $\| \sum y^n \| \sim (\sum ||y^n||^p)^{1/p}$, as in the [Kadets and Pełczyński, 1961/62] argument.

To achieve this we need a definition and a sublemma.

**Definition.** $A \subseteq L_p$ is $p$-uniformly integrable if $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall m(E) < \delta \ \forall z \in A$, we have $\int_E |z|^p < \varepsilon$.

**Sublemma.** Let $Y \subseteq L_p$, $2 < p < \infty$, with $Y \sim \ell_2$. There exists $Z \subseteq Y$ with $S_Z$ $p$-uniformly integrable.

This is proved in two steps. First showing a normalized martingale difference sequence $(x_n)$ with $\{(x_n)\}$ $p$-uniformly integrable has $A = \{\sum a_i x_i : \sum a_i^2 \leq 1\}$ also $p$-uniformly integrable by a stopping time argument.

The general case is to use the subsequence splitting lemma to write a subsequence of an $\ell_2$ basis as $x_i = y_i + z_i$ where the $(y_i)$ are a $p$-uniformly integrable (perturbation of) a martingale difference sequence and the $z_i$’s are disjointly supported and then use an averaging argument to get a block basis where the $z_i$’s disappear. □
Now we return to condition d) in Theorem 10 and recall by [Johnson, Maurey, Schechtman, and Tzafriri, 1979] every weakly null sequence in \( S_X \) has a subsequence \((x_i)\) with for some \( w \in [0, 1] \),

\[
\left\| \sum a_i x_i \right\| \overset{D_p}{\sim} \left( \sum |a_i|^p \right)^{1/p} \vee w \left( \sum |a_i|^2 \right)^{1/2}
\]

and d) asserts that for some absolute \( C \), \( w \overset{C}{\sim} \lim_i \|x_i\|_2 \).

Now clearly we can assume that \( w \geq \lim_i \|x_i\|_2 \) and if d) fails we can use this to construct our \( \ell_2 \)'s satisfying the lemma and thus obtain \((\sum \ell_2)_p \hookrightarrow X\). \( \square \)

So we have the dichotomy for \( X \subseteq L_p, 2 < p < \infty \).

Either

- \( X \hookrightarrow \ell_p \oplus \ell_2 \) or
- \((\sum \ell_2)_p \hookrightarrow X\).

In the latter case using \( L_p \) is stable we can get for all \( \varepsilon > 0 \),

\((\sum \ell_2)_p^{1+\varepsilon} \hookrightarrow X\). In fact we can get \((\sum \ell_2)_p\) complemented in \( X \) via
**Proposition 13.** For all $n$ let $(y^n_i)_i$ be a normalized basic sequence in $L_p$, $2 < p < \infty$, which is $K$-equivalent to the unit vector basis of $\ell_2$ and so that for $y_n \in [(y^n_i)_i]$, 
\[ \left\| \sum y_n \right\| \overset{K}{\sim} \left( \sum \|y_n\|_p \right)^{1/p}. \]

Then there exists subsequences $(x^n_i)_i \subseteq (y^n_i)_i$, for each $n$, so that $[\{x^n_i : n, i \in \mathbb{N}\}]$ is complemented in $L_p$.

**Proof.** By [Pełczyński and Rosenthal, 1974/75] each $[(y^n_i)_i]$ is $C(K)$-complemented in $L_p$ via projections 
\[ P_n = \sum_m y_m^*(x)y_m^n. \]

Passing to a subsequence and using a diagonal argument and perturbing we may assume there exists a blocking $(H^n_m)$ of $(h_i)$, in some order over all $n, m$, so that for all $n, m$, supp$(y_m^n)$, supp$(y_m^n) \subseteq H^n_m$. This uses $y_m^n \overset{w}{\to} 0$ and $y_m^n \overset{w}{\to} 0$ (in $L_p'$) as $m \to \infty$ for each $n$. Set 
\[ Py = \sum_{n,m} y_m^n*(y)y_m^n. \] We show $P$ is bounded, hence a projection onto a copy of $\left(\sum \ell_2\right)_p$. 


Let \( y = \sum_{n,m} y(n, m), y(n, m) \in H_n \).

\[
\|Py\| = \left\| \sum_n \sum_m y_n^*(y(n, m))y_m^n \right\|
\]

\[
\sim \left( \sum_n \left( \sum_m |y_m^*(y(n, m)|^2 \right)^{p/2} \right)^{1/p}.
\]

Now

\[
\left( \sum_m |y_m^*(y(n, m)|^2 \right)^{1/2} \sim \|Pny(n)\| \leq C(K)\|y(n)\|
\]

where \( y(n) = \sum_m y(n, m) \). So

\[
\|Py\| \leq \tilde{C}(K) \left( \sum \|y_n\|^p \right)^{1/p} \leq \tilde{C}(K)\|y\|.
\]

Remarks. The proof of Proposition 13 above is due to Schechtman. He also proved by a different much more complicated argument that the proposition extends to \( 1 < p < 2 \).

In [Haydon, Odell, and Schlumprecht] the proofs of all the results are also considered using Aldous’ (1981) theory of random measures. We are able to show if \((\ell_2)_p \hookrightarrow X \subseteq L_p, 2 < p < \infty, \) then given \( \varepsilon > 0 \) there exists \((Y_n)_p^{1+\varepsilon} \hookrightarrow X, \)
\( d(Y_n, \ell_2) < 1 + \varepsilon \) and moreover: there exist disjoint sets \( A_n \subseteq [0, 1] \) with for all \( n, y \in Y_n, \|y|_{A_n}\| \geq (1 - \varepsilon 2^{-n})\|y\| \)
and \([Y_n : n \in \mathbb{N}]\) is \((1 + \varepsilon) \cdot C_p^{-1}\) complemented in \(L_p\) where \(C_p\) is the norm of a symmetric normalized Gaussian random variable in \(L_p\). This is best possible by [Gordon, Lewis, and Retherford, 1973].

We can also deduce the [Johnson and Odell, 1981] result: \( X \subseteq L_p, 2 < p < \infty \), and \( X \) is a quotient of a subspace of \( \ell_p \oplus \ell_2 \Rightarrow X \hookrightarrow \ell_p \oplus \ell_2 \), by showing that such an \( X \) cannot contain \((\sum \ell_2)_p\).

The [Kadets and Pełczyński, 1961/62], [Johnson and Odell, 1974] results yield for \( X \subseteq L_p, 2 < p < \infty \)

- \( X \) is asymptotic \( \ell_p \Rightarrow X \hookrightarrow \ell_p \)
- \( X \) is asymptotic \( \ell_2 \Rightarrow X \hookrightarrow \ell_2 \).

**Definition.** \( X \) is asymptotically \( \ell_p \oplus \ell_2 \) if \( \exists K \forall n \forall (e_i)_1^n \in \{X\}_n \exists (w_i)_1^n \) with

\[
\left\| \sum_{i=1}^{n} a_i e_i \right\| \sim K \left( \sum_{i=1}^{n} |a_i|^p \right)^{1/p} \vee \left( \sum_{i=1}^{n} w_i^2 a_i^2 \right)^{1/2}.
\]
**Proposition 14.** Let $X \subseteq L_p$, $2 < p < \infty$. $X$ is asymptotically $\ell_p \oplus \ell_2$ iff $X \hookrightarrow \ell_p \oplus \ell_2$.

This follows easily from our results by showing that $(\sum \ell_2)_p$ is not asymptotically $\ell_p \oplus \ell_2$.

**Problem.** Let $X \subseteq L_p$, $p > 2$. Give an intrinsic characterization of when $X \hookrightarrow (\sum \ell_2)_p$. 